MACHAKOS UNIVERSITY

University Examinations for 2024/2025 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

SECOND YEAR FIRST SEMESTER EXAMINATION FOR DEGREE IN BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE BACHELOR OF SCIENCE IN ANALYTICAL CHEMISTRY BACHELOR OF EDUCATION (SCIENCE)

SMA 406: FUNCTIONAL ANALYSIS

DATE: 17TH JANUARY, 2025 **TIME: 8.30 AM – 10.30 AM**

INSTRUCTIONS:
Answer question ***ONE (Compulsory)*** *and* ***any other TWO*** *questions*

QUESTION ONE *(COMPULSORY)* (30 MARKS)

a) Show that if X is a Hilbert space, then
   i. $S, T \in L(X), (S+T)^* = S^* + T^*$ [3marks]
   ii. $S, T \in L(X), (T^*)^* = T$ [2marks]

b) Show that the mapping $T: C[a,b] \to K$ given by $Tf = f(\frac{a+b}{2})$ for all $f \in C[a,b]$ is a linear transformation [3marks]

c) Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in a normed space X, then prove that the weak limit x of $(x_n)_{n\in\mathbb{N}}$ is unique [3marks]

d) Show that $Re\,(\langle x,y\rangle) = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$ and $Im\,(\langle x,y\rangle) = Re\,(\langle x,iy\rangle)$ [5marks]

e) Explain what is a norm and give two examples of normed spaces [5marks]

f) Show that the function space $C[a,b]$ is a vector space [4marks]

g) Let X be a normed space and let $x_0 \neq 0$ be any element of X, then prove that there exists a bounded linear functional $\overline{f}$ on X such that $||\overline{f}|| = 1, ||f||_z = ||x_0||$ [5marks]

QUESTION TWO (20 MARKS)

a) The set $L(X,Y)$ of bounded linear operators from normed spaces X and Y with addition and scalar multiplication defined by

$$(T+S)x = Tx + Sx, x \in X,$$
$$(\alpha T)x = \alpha Tx, x \in X, \alpha \in \kappa$$ is a vector space. Prove that the mapping $T \to ||T||$ is a norm on this space [7marks]

b) Let X be an n −dimensional vector space, then prove that any proper subspace Y of X has dimension less than n. [4marks]

c) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed space X, then prove that strong convergence implies weak convergence with the same limit [4marks]

d) Define the term linear independence [2marks]

e) Explain the continuity of functions between normed spaces [3marks]

QUESTION THREE (20 MARKS)

a) Let X be a normed space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X with limit x. Prove that $(||x_n||)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}$ [5marks]

b) Prove that if X is an inner product space, then for all $x, y \in X, |\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$ and $|\langle x,y\rangle|^2 = \langle x,x\rangle\langle y,y\rangle$ if x and y are linearly dependent. [6marks]

c) Show that if X is a Hilbert space then $S, T \in L(X), (ST)^* = T^*S^*$ [3marks]

d) Differentiate between orthogonal and orthonormal sets [2marks]

e) Prove that for a bounded linear operator, $||Tx|| \leq M||x||, \forall x \in X$, there that of all constant M, there is a smallest one, given by $||T|| = \sup_{||x|| \leq 1} ||Tx||$ [4marks]

QUESTION FOUR (20 MARKS)

a) Prove that the spaces l^p are Banach spaces for $1 \leq p \leq \infty$ [8 marks]

b) Prove that if f is a bounded linear functional on a subspace Z of a normed space X then there exists a bounded linear functional $\bar{f}$ on X which is an extension of f to X and $||\bar{f}||_X = ||f||_Z$ where $||\bar{f}||_X = \sup_{\substack{x \in X \\ ||x|| = \bar{1}}}$ and $|f||_z = \sup_{\substack{x \in X \\ ||x|| = 1}}$, $||f||_z = 0$ in the trivial case $z = \{0\}$ [8 marks]

c) Explain what is a Cauchy sequence and show that every convergent sequence is a Cauchy sequence [4marks]

QUESTION FIVE (20 MARKS)

a) State the Generalized Hann Banach theorem and prove it. [10marks]

b) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a Banach space and $s_n = x_1 + \ldots + x_n$. Prove that if $\sum_{n=1}^{\infty} ||x_n|| < \infty$ then the series $\sum_{n=1}^{\infty} x_n$ converges and if we denote $\lim_{n\to\infty} s_n$ by $\sum_{n=1}^{\infty} x_n$ then $||\sum_{n=1}^{\infty} x_n|| \leq \sum_{n=1}^{\infty} ||x_n||$ [5marks]

c) Let X and Y be normed spaces over $K(\mathbb{R} or \mathbb{C})$. Let $T: X \to Y$ be linear transformation. Prove that T is continuous at 0 and there exists a number m such that for all $x \in X, ||Tx|| \leq m||x||$ [5marks]