



MACHAKOS UNIVERSITY

University Examinations 2019/2020 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING

BACHELOR OF SCIENCE IN MATHEMATICS

BACHELOR OF EDUCATION

SMA 402: FIELD THEORY

DATE: 18/01/2021

TIME: 8.30-10.30 AM

INSTRUCTIONS TO CANDIDATES

Answer question ONE and any other TWO questions

QUESTION ONE 30 MARKS (COMPULSORY)

- a) Using relevant examples differentiate between a finite Field and a Field of characteristic zero. Give Two examples of each. (3 marks)
- b) Let $x = \sqrt{3 + \sqrt{5}}$. Find the minimal polynomial of x over $\mathbb{Q}$ (3 marks)
- c) Find the splitting field of the polynomial $P(x) = x^4 - 10x + 21 \in \mathbb{Q}[x]$. Hence identify the basis and dimension of the basis. (4 marks)
- d) Show that $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$ is a Field. (4 marks)
- e) Consider the polynomial in $p(x) = x^2 + x + 1$ in $\mathbb{Z}_2[x]$
 - i. Describe what is meant by an irreducible polynomial. (2 marks)
 - ii. Verify that the polynomial $p(x)$ is irreducible in $\mathbb{Z}_2$ (1 mark)
 - iii. Let $\mathbb{Z}_2[\alpha]$ be the extension field of $\mathbb{Z}_2$ where α is algebraic over $\mathbb{Z}_2$. In terms of α list and simplify all the elements of $\mathbb{Z}_2[\alpha]$. Construct addition and multiplication tables of $\mathbb{Z}_2[\alpha]$ then verify that $\mathbb{Z}_2[\alpha]$ is a field. (9 marks)

- iv. Find $\text{irr}(\alpha, Q)$ and $\text{deg}(\alpha, Q)$ for the given algebraic number $\alpha \in \mathbb{C}$ over Q
 where $\alpha = \sqrt{5} + i$ (3 marks)

QUESTION TWO (20 MARKS)

- a) Give definitions of the following terms giving examples in each:
- An algebraic element α of an extension field E of a field F . (2 marks)
 - A transcendental element α of an extension field E of a field F . (2 marks)
- b) Let $f(x) = x^4 - 8x^2 + 15$ Over the field Q .
- Find the zeros of the polynomial $f(x)$. (4 marks)
 - Find the splitting field F of the polynomial $f(x)$. (2 marks)
 - Exhibit a tower of fields in (ii) above (2 marks)
 - Find the index of F in Q . (2 marks)
- c)
 - Describe what is meant by intersection of subfields: Give an example. (3 marks)
 - Prove that $x^2 - 3$ is irreducible over $Q(\sqrt[3]{2})$ (3 marks)

QUESTION THREE (20 MARKS)

- a)
 - Show that the polynomial $x^2 + 1$ is irreducible in $Z_3[x]$ (3 marks)
 - Let β be a zero of the polynomials $x^2 + 1$ in the extension field $Z_3[\beta]$.
 List down the nine elements of $Z_3[\beta]$. Give the addition and multiplication tables. Justify that $Z_3[\beta]$ is the finite field $GF(3^2)$ (8 marks)
- b) Show that a field F is algebraically closed if and only if every non constant polynomial in $F[x]$ factors in $F[x]$ into linear factors. (5 marks)
- c) Find the degree and the basis for the field extension $Q(\sqrt[3]{2}, \sqrt{3})$ over Q . Hence construct the tower of fields clearly indicating all the subfields. (4 marks).

QUESTION FOUR (20 MARKS)

- a) Find a basis for the extension field $Q(\sqrt{5} + i)$ over Q (3 marks)
- b) Show that if E is a finite extension field of a field F and K is a finite extension field of E then K is a finite extension of F and $[K:F] = [K:E][E:F]$ (7 marks)

- c) Is the polynomial $p(x) = x^3 + 2x + 3$ in $Z_5[x]$ irreducible? Why? Express it as a product of irreducible polynomials in $Z_5[x]$ (5 marks)
- d) Find all the generators of the cyclic multiplicative group of units of the field Z_{11} (5 marks)

QUESTION FIVE (20 MARKS)

- a) i) Show that it is impossible by straight edge and compass alone to trisect an angle of 60° (4 marks)
- ii) Show that it is impossible by straight edge and compass alone to duplicate the cube. (4 marks)
- b) Write $F_3 = \{0,1,2\}$, the finite field of order 3 with arithmetic performed modulo 3. Let $\alpha = \sqrt{2}$.
- i) Is $2 + \alpha$ a square in $F_3(\alpha)$. (3 marks)
- ii) Let $\beta = \sqrt{2 + \sqrt{2}}$, calculate the degree of the extension $F_3(\beta):F_3(\alpha)$ (3 marks)
- iii) Calculate the degree of the extension $F_3(\beta):F_3$ (3 marks)
- iv) Justify that the set $\{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in Q\}$ is a field. (3marks)