MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECOND YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN MATHEMATICS

SMA 204: ALGEBRAIC STRUCTURES

DATE: 24/3/2021 **TIME: 8.30-10.30 AM**

INSTRUCTIONS TO CANDIDATES

Answer Question One and any other two questions

QUESTION ONE (30 MARKS)

a) Explain the meaning of the following terms

(i) a commutative binary operation on a set A, (2 marks)

(ii) an associative binary operation on a set A. (2 marks)

b) Let $\mathbb{R} \setminus \{0\}$ be the set of all non-zero real numbers and let $\times$ be the usual multiplication of real numbers. Prove that $(\mathbb{R} \setminus \{0\}, \times)$ is a group. (5 marks)

c) Define a relation $\sim$ on the set $\mathbb{Z}$ as follows: $a \sim b$ if and only if $a - b$ is a multiple of three. Determine if $\sim$ is an equivalence relation. (5 marks)

d) Let $\oplus$ denote operation of addition modulo 5 on the set $\mathbb{Z}_5$. Construct the Cayley table for the operation $\oplus$ on $\mathbb{Z}_5$. (5 marks)

e) Consider $\mathbb{Z}_{20}$ under multiplication modulo 20 (denoted by $\otimes$).

(i) Is $\langle \mathbb{Z}_{20}, \otimes \rangle$ a group? Justify your answer. (3 marks)

(ii) Determine the elements of U_{20} and construct the corresponding Cayley table. (4 marks)

(iii) Let H be the cyclic subgroup of U_{20} generated by $13 \in U_{20}$. Determine the elements of H. (4 marks)

QUESTION TWO (20 MARKS)

a) Let $\odot$ be a binary operation defined on $\mathbb{Z}$ by $a \odot b = ab - a - b$.

(i) Show that $\odot$ is a commutative binary operation. (2 marks)

(ii) Is $\odot$ an associative binary operation? Justify your answer. (2 marks)

b) Let $\mathbb{Q}$ be the group of rational numbers under standard addition. Let F_7 be the subset of $\mathbb{Q}$ given by

$$F_7 = \left\{x \in \mathbb{Q} | x = \frac{m}{7} for\ some\ m \in \mathbb{Z}\right\}.$$

Determine whether F_7 is a subgroup of $\mathbb{Q}$. (5 marks)

c) Suppose that is a group and $a \in G$ be an element such that $a^2 = a$. Prove that $a = e$ where e is the identity element in G. (2 marks)

d) Let $M_2(\mathbb{Z})$ denote the group of all 2×2 matrices with integer entries under matrix addition. Let T be the subset of $M_2(\mathbb{Z})$ given by

$$T = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in M_2(\mathbb{Z}) such\ that\ a, b\ and\ c\ are\ even\ numbers\right\}.$$

Show that T is a subgroup of $M_2(\mathbb{Z})$. (5 marks)

QUESTION THREE (20 MARKS)

a) State the Lagrange's theorem. (2 marks)

b) Let $G = \langle \mathbb{Z}, + \rangle$ be the group of integers under the standard addition.

(i) Let K be the subset consisting of all odd numbers in $\mathbb{Z}$. Is K a subgroup of G? Justify. (2 marks)

(ii) Let H be the subset consisting of all even numbers in $\mathbb{Z}$. Prove that H is a subgroup of G. (3 marks)

(iii) Determine all the distinct left cosets of H. (3 marks)

c) Let G be a group

(i) Show that if $a, b \in G$ are elements such that $a^2b^2 = a^4b$ then $ba = a^3$. (3 marks)

(ii) Prove that if for all $a, b \in G$ then G is a commutative group. (3 marks)

d) Suppose that G is a group of order 105 and suppose that H is a non-trivial subgroup of G. What are the possible values of $o(H)$? Justify your answer. (4 marks)

QUESTION FOUR (20 MARKS)

a) Explain the meaning of the following terms:

(i) a *right coset* of a subgroup, (2 marks)

(ii) the *kernel* of a group homomorphism. (2 marks)

b) Let $U_{14} = \{1,3,5,9,11,13\}$ be the group of invertible elements in $\mathbb{Z}_{14}$ under multiplication modulo 14. Let H be the cyclic subgroup of U_{14} generated by 13. Determine all the distinct left cosets of H. (6 marks)

c) Let $G_1 = \langle \mathbb{R}, + \rangle$ be the group of real numbers under standard addition of real numbers and $G_2 = \langle \mathbb{R}^{+,\times} \rangle$ be the group of all positive real numbers under standard multiplication. Show that

(i) the function $f: G_1 \to G_2$ defined by $f(x) = exp(x)$ for any $x \in G_1$ is a group homomorphism. (Here, $exp(x) = e^x$ is the usual exponential function). (5 marks)

(ii) the function $g: G_2 \to G_1$ defined by $g(a) = ln(a)$ for any $a \in G_2$ is a group homomorphism. (Here $ln(a)$ is the usual natural logarithm function). (5 marks)

QUESTION FIVE (20 MARKS)

a) Explain the meaning of the following terms:

i. an *integral domain*, (2 marks)

ii. the *characteristic* of a ring, (2 marks)

b) Suppose that R is a set endowed with an additive binary $+$ and a multiplicative binary operation denoted by $\times$. What properties must the triple $(R, +, \times)$ satisfy in order for it to be a ring? (8 marks)

c) Let $\mathbb{Q}$ denote the set of rational numbers. Define the set

$$\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$$

If we define addition and multiplication on $\mathbb{Q}(\sqrt{3})$ as follows

$$(a + b\sqrt{3}) \oplus (c + d\sqrt{3}) = (a + c) + (b + d)\sqrt{3}$$
$$(a + b\sqrt{3}) \otimes (c + d\sqrt{3}) = (a + b\sqrt{3})(c + d\sqrt{3})$$

Show that $(\mathbb{Q}(\sqrt{3}), \oplus, \otimes)$ is a ring. (8 marks)