MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (PROGRAMMING AND STATISTICS)

BACHELOR OF ECONOMICS AND STATISTICS

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION (SCIENCE)

BACHELOR OF EDUCATION (ARTS)

SMA 403: GALOIS THEORY

DATE: 17/12/2021 **TIME: 11.00-1.00 PM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (30 MARKS) (COMPULSORY)

a) Let Ebe a field extension of a field F. Define the Galois group $G(E/F)$. (3 marks)

b) Briefly state the fundamental theorem of Galois theory. (3 marks)

c) Prove that the set of all automorphism of a field F is a group under composition of functions. (4 marks)

d) Let Fbe a field of characteristic different from 2.

i. Show that if $f(x) = ax^4 + bx^2 + c$ where $a \neq 0$, then $f(x)$ is solvable by radicals over F . (6 marks)

ii. If in part (i) $a = 0.5$, $b = 3.5$, $c = 5$determine the Galois group of $f(x)$ over F. (4 marks)

e) Let $\{\alpha_i : i \in \mathbb{N}\}$ be a collection of automorphism of a field E, show that $E(\alpha_i) = \{t \in E : \alpha_i(t) = t \forall \alpha_i\}$ is a subfield of E. (5 marks)

f) Define what is meant by a primitive element of a Field extension $K$of F, hence identify the primitive element of the extension $K = \mathbb{Q}(\sqrt{7}, i)$ of $\mathbb{Q}$. Determine the minimal monic polynomial associated with the primitive element you have identified. (5 marks)

QUESTION TWO (20 MARKS)

a) Let $f(x) = x^4 - x^2 - 2$ over $\mathbb{Q}$.

i. Find the zeros of the polynomial $f(x)$. (4 marks)

ii. Find the splitting field E of the polynomial$f(x)$. (3 marks)

iii. Find the index of E in $\mathbb{Q}$. (2 marks)

iv. Construct the Galois group $G(E/\mathbb{Q})$ (4 marks)

b) By using (a) above construct lattice diagrams of the subgroups of Galois group and subfields of the splitting field. (4 marks)

c) Identify with reasons a subgroup of the symmetric group $S_n$isomorphic to $G(E/\mathbb{Q})$.

(3 marks)

QUESTION THREE (20 MARKS)

a) Let$E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{3})$. Consider the Galois group $G(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q})$. The field $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{3})$ may be considered as a vector space. Identify the elements of the basis B of this vector space and state its dimension. (6 marks)

b) Let $\tau_1, \tau_2, \tau_3 \in G(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q})$such that $\tau_2 : \sqrt{2} \mapsto -\sqrt{2}, \tau_3 : \sqrt{3} \mapsto -\sqrt{3}, \tau_5 : \sqrt{5} \mapsto -\sqrt{5}$. Compute the following indicated elements of E.

i. $\tau_5^2 \tau_3 \tau_2 (\sqrt{2} + \sqrt{45})$ (3 marks)

ii. $\tau_5 \tau_3 \tau_2 \left(\frac{\sqrt{2} - 3\sqrt{5}}{2\sqrt{3} - \sqrt{2}}\right)$ (3 marks)

c) Clearly identify and describe elements of Galois group $G(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q})$. (8 marks)

QUESTION FOUR (20 MARKS)

a) Define what is meant by a solvable group G, hence show that the Symmetric group S_4 is solvable. (4 marks)

b) Use Cardan's Formula to solve the degree three polynomial equation $x^3 + x^2 - 3x + 2 = 0$.

(6 marks)

c) Let K be a field extension of H and $h(x)$ be a polynomial in $H(x)$. Show that any automorphism in $G(K/H)$ defines a permutation of the roots of $h(x)$ that lie in K. (4 marks)

d) Define what is meant by a cyclotomic extension for any field E. Hence or otherwise write down the splitting field of the polynomial $x^5 - 1 \in \mathbb{Q}[x]$ and its corresponding Galois Group. (6 marks)

QUESTION FIVE (20 MARKS)

a) Give definitions of the following terms:

i. Separable extension of a field F (2 marks)

ii. Normal extension of a field F. (2 marks)

b) State the symmetric function theorem, hence express the following symmetric functions in elementary symmetric functions in x_1, x_2, x_3: (2 marks)

i. $x_1^2 + x_2^2 + x_3^2$ (3 marks)

ii. $x_1^3 + x_2^3 + x_3^3$ (3 marks)

c) Prove that the symmetric group S_n is not solvable for $n \geq 5$. (8 marks)