MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FIRST YEAR FIRST SEMESTER EXAMINATION FOR

MASTER OF SCIENCE (MATHEMATICAL MODELLING AND COMPUTATIONAL)

SMA 800: FUNCTIONAL ANALYSIS

DATE: 22/6/2021 **TIME: 2:00 – 5:00 PM**

INSTRUCTIONS TO CANDIDATES

Answer ALL the questions in Section A and ANY TWO Questions in Section B

SECTION A (COMPULSORY)

QUESTION ONE (20 MARKS)

a) Let $X = \{x \in \mathbb{R} | x \geq 1\} \subset \mathbb{R}$ and let the mapping $T: X \to X$ be defined by $Tx = \frac{x}{2+x^{-1}}$.
Show that T is a contraction and find the smallest $\propto$ (5 marks)

b) Prove that the space $C[a, b]$ is not an inner product space and hence not Hilbert.
(5 marks)

c) Show that $p(y) = |y|$ is a semi-norm on $\mathbb{C}$. (5 marks)

d) Show that $Ts = 3s - 1$ is not a linear operator. (5 marks)

SECTION B: ANSWER ANY TWO QUESTIONS

QUESTION TWO (20 MARKS)

a) Prove that $Ty = \frac{1}{y}$ is not uniformly continuous on the interval (0,1). (5 marks)

b) Prove that $T: X \to Y$, be defined by $T(x) = 4x$, is linear. (5 marks)

c) Find the fixed points of the mapping $T: \mathbb{R} \to \mathbb{R}$ defined by $Tx = 3x^2 - 4$. (5 marks)

d) Take $X = (0, 1]$ as a sequence of $\mathbb{R}$. Then X is not complete. (5 marks)

QUESTION THREE (20 MARKS)

a) Show that $T: C[a,b] \rightarrow C[a,b]$ defined by $Tx(t) = \int_a^b x(t)dt \quad t \in [a,b]$ is a linear operator on $C[a,b]$ (5 marks)

b) Let T be a bounded linear operator, then

i) $x_n \rightarrow x$ [where $x_n, x \in D(T)$] implies $T: x_n \rightarrow Tx,$ (5 marks)

ii) The kernel of T is closed. (5 marks)

c) Prove that if Y is a banach space then $B(X,Y)$ is a banach space. (5 marks)

QUESTION FOUR (20 MARKS)

a) Prove that if X is any non-empty set and define $d: XxX \rightarrow \mathbb{R}^+ \cup \{0\}$ by

$d(x,y) = \{ \begin{matrix} 1, x \neq y \\ 0, x = y \end{matrix}$ then d is a metric on X.

b) Show that $\int_a^b x(t)dt,\ x \in c[a,b]$ is a linear and bounded function on $c[a,b]$. (5 marks)

c) Prove that if $T: X \rightarrow X$, where X is a normed linear space is continuous at the origin, then T is uniformly continuous. (5 marks)

d) Let E be an inner product space. Then for arbitrary $x, y \in E$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$ (5 marks)

QUESTION FIVE (20 MARKS)

a) Prove that if X and Y are Banach spaces and $T: T: X \rightarrow Y, D(T) \rightarrow Y$, is a closed linear operator , where $D(T) \complement X$. Then $D(T)$ is closed in X, the operator T is bounded.

(5 marks)

b) Prove that in the set $\mathbb{R}$ of all real numbers define d on $\mathbb{R}x\mathbb{R}$ by

$d(x,y) = \sum|x-y| \ \forall\, x,y \in \mathbb{R}$, d is a metric space. (5 marks)

c) A compact subset M of a metric space is closed and bounded. (5 marks)

d) Show that $\|y\| = (\sum_1^\infty |y_n|^p)^{\frac{1}{p}} \geq 0$ is a norm in l_p. (5 marks)