MACHAKOS UNIVERSITY

University Examinations for 2024/2025 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

THIRD YEAR FIRST SEMESTER EXAMINATION FOR DEGREE IN
BACHELOR OF SCIENCE (MATHEMATICS AND STATISTICS)
BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)
BACHELOR OF ECONOMICS AND STATISTICS
BACHELOR OF SCIENCE (MATHEMATICS)
BACHELOF EDUCATION (SCIENCE)
BACHELOR OF EDUCATION (ARTS)
BACHELOR OF ARTS

SMA 300: REAL ANALYSIS I

DATE: 15TH JANUARY, 2025 **TIME: 2.00 PM – 4.00 PM**

INSTRUCTIONS:
Answer question ***ONE (Compulsory)*** *and* ***any other TWO*** *questions*

QUESTION ONE *(COMPULSORY)* (30 MARKS)

a) Let A and B be neighborhoods of a point x. Show that the intersection $A \cap B$ is a neighborhood of x **[3Marks]**

b) Use ratio test to determine the convergence of

$$\sum_{n=2}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$ **[4Marks]**

c) Prove that a set $X$is closed iff its complement is open **[5Marks]**

d) Determine and prove whether the set of all integers is countable. **[3Marks]**

e) Describe what is meant by the statement "The set $\mathbb{R}$ of real numbers is complete". Exhibit an example to illustrate why the set $\mathbb{Q}$ of rational numbers is not complete. **[3Marks]**

f) Explain the following terms

i. An upper bound,

ii. Infimum of a set

iii. Countable set. **[3Marks]**

g) Determine whether the following sequence is monotonic and/or bounded. If bounded determine its supremum and infimum.

$$\left\{\frac{2}{n^2}\right\}_{n=5}^{\infty}$$ **[5Marks]**

h) Let (S_n) be a sequence of real numbers. If $\lim_{n\to\infty} S_n = l_1$ and $\lim_{n\to\infty} S_n = l_2$ then prove $l_1 = l_2$ **[4Marks]**

QUESTION TWO (20 MARKS)

a) Test for convergence of the following series

(i) $\sum_{n=1}^{\infty} \frac{10^n}{n4^{2n+1}}$ (ii) $\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ (iii) $\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}$ **[9 Marks]**

b) Show that from the first principle that $\lim_{x\to 2}(x^2 - 4x + 2) = -2$ **[4 Marks]**

c) Determine whether the function $f(x) = \begin{cases} -x^2 + 4, & x \le 3 \\ 4x - 8, & x > 3 \end{cases}$ is continuous at $x = 3$. Justify your answer. **[4 Marks]**

d) Let $A_n = (2 - \frac{1}{2n}, 5 + \frac{1}{n})$ $n \in \mathbb{N}$. Find:

(i) $\bigcup_{n=1}^{\infty} A_n$ (ii) $\bigcap_{n=1}^{\infty} A_n$ **[3Marks]**

QUESTION THREE (20 MARKS)

a) Prove that the empty set $\emptyset$ is open **[4Marks]**

b) Show that if $p \in A$ and $p \notin A'$, then p is an isolated point . **[4Marks]**

c) Prove that every convergent sequence of real numbers is bounded. **[4Marks]**

d) Determine the convergence of $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$ **[4Marks]**

e) For each positive integer k, let $\mathbb{R}^k$ be the set of all k-tuples where $X = \{x_1, \ldots x_k\}$. Define a function d by $d(x, y) = \sum_{i=1}^{k} |x_i - y_i| \; \forall x, y \in \mathbb{R}$. Show that d is a metric space. **[4Marks]**

QUESTION FOUR (20 MARKS)

a) Prove that convergence of a sequence implies boundedness. Use an example to show that the converse need not be true. **[5Marks]**

b) Prove that the set of rational numbers $\mathbb{Q}$ has no interior point. **[4Marks]**

c) Consider the set $A = (1,10) \cup \{11\}$ .Show that 11 is the isolated point of A and both 1 and 10 are the limit points of A. **[5Marks]**

d) Let $X = \mathbb{R}$. Describe what is meant by

i) Neighborhood of a point $x \in X$. **[2Marks]**

ii) An open set $O \in X$. **[2Marks]**

e) Given that $A = (3,7)$ find the exterior and the boundary of A. **[2Marks]**

QUESTION FIVE (20 MARKS)

a) For the set $A = (-\infty, 4) \cup \{5,9\} \cup [6,7)$. Find

i. Interior of A.

ii. Exterior of A. **[4Marks]**

b) Determine the n^{th} partial sum of the series below. Hence test for its convergence.

$\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$ **[5Marks]**

c) Show that $f(x) = 2x + 1$ is Riemann integrable on $[1,2]$ and hence find $\int_1^2 (2x + 1)dx$, **[8Marks]**

d) Show that a finite intersection of open sets is open. **[3Marks]**