



MACHAKOS UNIVERSITY

UNIVERSITY SUPPLEMENTARY EXAMINATIONS

**FOURTH YEAR SECOND SEMESTER EXAMINATIONS FOR THE DEGREE OF
BACHELOR OF SCIENCE IN (STATISTICS AND PROGRAMMING ,
MATHEMATICS AND COMPUTER SCIENCE) BACHELOR OF ARTS IN
MATHEMATICS AND ECONOMICS, BACHELOR OF EDUCATION (SCIENCE,
ARTS)**

SMA 465: MEASURE AND PROBABILITY.

DATE : _____ TIME: 2 HOURS

INSTRUCTIONS: Attempt Question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

- a. Define the set Ω , outlining any three basic operations on subsets of Ω .
(4 marks)
- b. Prove the assertions:
 - i. $\underline{\lim} A_n \subseteq \overline{\lim} A_n$. (1 mark)
 - ii. $(\underline{\lim} A_n)^c = \overline{\lim} A_n^c$. (3 marks)
- c. Show that the class I of all intervals of $\mathbb{R}$ is a semi-algebra. (6 marks)
- d. Explain any three examples of measure. (3 marks)

e. Let A be σ -algebra of subsets of Ω . Let m be an application from A to $\mathbb{R}_+$.

Suppose that we have: For all sequences $(A_n)_{n \geq 0}$ of pairwise disjoint elements of

A , the σ -additivity formula holds $m\left(\sum_{n \geq 0} A_n\right) = \sum_{n \geq 0} m(A_n)$. Show that the two

assertions are equivalent:

- $m(\emptyset) = 0$.
- m is a proper application constantly equal to $+\infty$, that is $\exists A \in A, m(A) < +\infty$.

Show why the results you obtain still holds if m is only finitely additive on an algebra. (3 marks)

f. Let Ω be a nonempty set and put $A = \{A \subset \Omega, A \text{ is countable or } A^c \text{ is countable}\}$.

i. Show that A is a σ -algebra. (4 marks)

ii. Let $\Omega = [0, 1]$. Show that $\left[0, \frac{1}{2}\right]$ is not measurable in the measurable space (Ω, A) . (2 marks)

g. Consider the Lebesgue measure λ on $\mathbb{R}$.

i. Show that for any $x \in \mathbb{R}$, $\lambda(\{x\}) = 0$. (3 marks)

ii. Show that for countable A subset of $\mathbb{R}$, $\lambda(\{A\}) = 0$. (1 mark)

QUESTION TWO (20 MARKS)

a. Generalize the formula $1_{A \cup B} = 1_A + 1_B - 1_{A \cap B}$ to any countable union. Precisely by applying it to two sets and next to three sets, to obtain the general formula.

$$1_{\bigcup_{1 \leq j \leq n} A_j} = \sum_{1 \leq j \leq n} 1_{A_j} + \sum_{r=2}^n (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} 1_{A_{i_1} \dots A_{i_r}}. \quad (9 \text{ marks})$$

b. Show that the class of rectangles S is a semi-algebra. Extend this result to a product of k measurable spaces, $k \geq 2$. (4 marks)

- c. Let Ω be a non-empty set and set $A = \{A \subset \Omega, A \text{ is finite or } A^c \text{ is finite}\}$
- Show that A is an algebra. (3 marks)
 - Show that A is a σ -algebra if Ω is finite. (1 mark)
 - If Ω is infinite, consider an infinite subset $C = \{w_0, w_1, w_2, \dots\}$ and set $A_n = \{w_{2n}\}, n \geq 0$. Show that the A_n are in A and their union is not. (2 marks)
 - Conclude that A is a σ -algebra if and only if Ω is finite. (1 mark)

QUESTION THREE (20 MARKS)

- a. Prove the assertions:
- $\left(\bigcup_{n \geq 0} A_n\right) \setminus \left(\bigcup_{n \geq 0} B_n\right) \subseteq \bigcup_{n \geq 0} (A_n \setminus B_n)$ (4 marks)
 - $\left(\bigcup_{n \geq 0} A_n\right) \Delta \left(\bigcup_{n \geq 0} B_n\right) \subseteq \bigcup_{n \geq 0} (A_n \Delta B_n)$ (3 marks)
- b. Let $A_n, n \geq 0$ be a sequence of subsets of Ω . Set $B_0 = A_0, B_1 = A_0^c A_1, B_2 = A_0^c A_1^c A_2, \dots, B_k = A_0^c A_1^c \dots A_{k-1}^c A_k$ for $k \geq 2$.
- Show that $\bigcup_{n \geq 0} A_n = \sum_{k \geq 0} B_k$. (7 marks)
 - Interpret the meaning of k for $w \in B_k$, for any $w \in \bigcup_{n \geq 0} A_n$. (2 marks)
- c. Let $\Omega = \Omega_1 \times \Omega_2$ be a non-empty product defined by $\Omega = \Omega_1 \times \Omega_2 = \{(w_1, w_2) \in \Omega, w_1 \in \Omega_1 \text{ and } w_2 \in \Omega_2\}$. Define a rectangle $A \times B = \{(w_1, w_2) \in \Omega, w_1 \in A \text{ and } w_2 \in B\}$, where $A \subseteq \Omega_1, B \subseteq \Omega_2$.
- Show that the intersection of a rectangle is a rectangle. (2 marks)
 - Let $A \times B$ be a rectangle. Show that $(A \times B)^c = A^c \times B + A \times B^c + A^c \times B^c$. (2 marks)

QUESTION FOUR (20 MARKS)

- a. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ be a measure space. Show that if m is translation-invariant and $m([0,1]) = 1$ then m is the Lebesgue measure λ . (6 marks)
- b. Let $(\Omega, \mathcal{A}, m)$ be measure space. Show the following properties:
- m is non-decreasing: if $A \subset B$, then $m(A) \leq m(B)$. Besides, show, if $m(B)$ is finite, then $m(B \setminus A) = m(B) - m(A \cap B)$. (2 marks)
 - A measure m finite if and only if $m(\Omega)$ is finite if and only if m is bounded. (2 marks)
 - A measure m is σ -sub-additive. In a similar manner show that if $m(\emptyset) = 0$, and m is non-negative and additive, then it is sub-additive. (2 marks)
 - Show that m is continuous in the following sense:
 - Let $(A_n)_{n \geq 0}$ be a non-decreasing sequence of elements of $\mathcal{A}$ and set $A = \bigcup_{n \geq 0} A_n$. Then $m(A_n)$ increases to $m(A)$ as $n \rightarrow +\infty$. (6 marks)
 - Let $(A_n)_{n \geq 0}$ be a non-increasing sequence of elements of $\mathcal{A}$ and set $A = \bigcap_{n \geq 0} A_n$ such that $m(A_{n_0}) < \infty$ for some $n_0 \geq 0$. Then $m(A_n)$ decreases to $m(A)$ as $n \rightarrow +\infty$. (2 marks)

QUESTION FIVE (20 MARKS)

- a. Prove the algebra generated by a semi-algebra S is the class of all finite sums of its elements. $a(S) = \{A_1 + A_2 + \dots + A_k, A_i \in S, k \geq 1\}$. (7 marks)
- b. Let C be an algebra and a monotone class. Show that C is a sigma-algebra. (5 marks)
- c. Let D be a π -class and a Dynkin system. Show that D is a σ -algebra. (4 marks)

h. Consider a Lebesgue measure λ_F on $\mathbb{R}$ where F is a distribution function.

Show that for any $x \in \mathbb{R}$, $\lambda_F(\{x\}) = F(x) - F(x-0)$, where $F(x-0)$ is the left-hand limit of F at x . Deduce from this that $\lambda_F(\{x\}) = 0$ if and only if F is continuous at x .

(4 marks)