MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FIRST YEAR FIRST SEMESTER EXAMINATION FOR

MASTER OF SCIENCE (MATHEMATICAL MODELLING AND COMPUTATION)

SMA 804: MODELING WITH ORDINARY DIFFERENTIAL EQUATIONS

DATE: 24/5/2019 **TIME: 2:00 – 5:00 PM**

INSTRUCTIONS TO CANDIDATES

QUESTION ONE (COMPULSORY) (20MARKS)

a) State any four steps of mathematical modeling (4 marks)

b) The non-dimensional Lokta-Volterra model is given by $\begin{matrix} \dot{u} = u(1-u) \\ \dot{v} = \alpha v(u-1) \end{matrix}$ where α is a positive constant. Given that the phase trajectories for this model are $\alpha u + v - \ln(u^{\alpha} v) = C$ where C is a constant. Sketch the phase trajectories on the uv phase plane and briefly explain the dynamics (6 marks)

c) In aid of a suitable flow diagram, design basic SEIR model equations to describe the transmission of malaria from mosquito population to human population. (6 marks)

d) Using the Liapunov test function $V(x, y) = x^2 + 2y^2 = a,\ a > 0$, show that the zero solution of the system $\dot{x} = -x - 2y^2,\ \dot{y} = xy - y^3$ is uniformly and asymptotically stable.

(4 marks)

QUESTION TWO (20 MARKS)

a) The spread of common cold/flu in a given confined population can be modeled using the SIR model:

$$\dot{S} = -\beta SI, \quad \beta > 0, S(0) > 0$$
$$\dot{I} = \beta SI - \sigma I, \sigma > 0, I(0) > 0$$
$$\dot{R} = \sigma I, \; R(0) = 0$$

i) Sketch the flow diagram used to deduce the above model equations. (4 marks)

ii) State any three assumptions used to develop the above SIR model that describes the spread of the common cold/flu. (3 marks)

iii) Realistically, the population size changes and common cold is known to recur i.e. a recovered individual can get the disease once again. Improve the given model equations by accounting for this realistic phenomenon. (3 marks)

b) The Lokta-Volterra model for two-species ecosystem is given by: $\begin{aligned}\dot{x} &= ax - cxy \\ \dot{y} &= -by + dxy\end{aligned}$ where a, b, c, d are positive constants.

i) Determine the equilibrium points of the model (4 marks)

ii) Determine the phase paths of this system of equations. (6 marks)

QUESTION THREE (20 MARKS)

a) In an electrical circuit the current $I(t)$ is modeled by $L\dfrac{dI}{dt} + RI = E(t)$. Determine the current in a circuit where the resistance is 12Ω, the inductance is $4H$, a battery gives a constant voltage of $60V$, and the switch is turned on when $t = 0$. (7 marks)

b) Let $\vec{x}^*(t)$ be a given solution vector of the non-autonomous system $\dfrac{d\vec{x}(t)}{dt} = \vec{X}(\vec{x}, t)$.

Define Liapunov stability of the solution $\vec{x}^*(t)$ to such a system (3 marks)

c) Given the dynamical system: $\begin{aligned}\dot{x} &= y(1 - x^2) \\ \dot{y} &= -x(1 - y^2)\end{aligned}$

i) Determine the equilibrium points (5 marks)

ii) Show that the phase paths are given by the equation $(1 - x^2)(1 - y^2) = C$, where C is a constant. (5 marks)

QUESTION FOUR (20 MARKS)

a) Vibrations of a two-storey shear building can be modeled by the system of equations: $\begin{aligned} m\ddot{x}_1 + 2kx_1 - kx_2 &= F_1(t) \\ m\ddot{x}_2 - kx_1 + kx_2 &= F_2(t) \end{aligned}$. Sketch the two-storey building indicating where the $m, k, x_1, x_2, F_1(t), F_2(t)$ lies in the diagram. (7 marks)

b) Consider the predator, $P(t)$ -prey, $N(t)$ model $\begin{aligned} \dot{N} &= N[a - bP] \\ \dot{P} &= P[cN - d] \end{aligned}$ where a, b, c, d are positive constants. Improve this model such that each of the species $P(t)$ and $N(t)$ have logistic growth in the absence of the other and state the meaning of each constant in the improved model. (8 marks)

c) One of the equations that is used to model vehicular traffic flow on a highway can be expressed in discrete form as $\rho(i,k+1)\Delta x - \rho(i,k)\Delta x = q(i-1,k)\Delta t - q(i,k)\Delta t$:

i) State the assumption and principle used to arrive at the above equation. (3 marks)

ii) Give the two instances that contribute to the change in traffic density at the time instance t_{k+1} (2 marks)

QUESTION FIVE (20 MARKS)

The SEIR model consists of the following models equations:

$$\begin{aligned} \dot{S} &= -\beta SI + \lambda - \mu S \\ \dot{E} &= \beta SI - (\mu + k)E \\ \dot{I} &= kE - (\gamma + \mu)I \\ \dot{R} &= \gamma I - \mu R \end{aligned}$$

a) Sketch the flow diagram of the above model equations. (5 marks)

b) State what each of the following parameters β, λ, μ, k, γ represents. (5 marks)

c) Obtain the matrices F and V^{-1} for this model, hence show that the reproduction number is given by $R_0 = \dfrac{k\beta\lambda}{\mu(k+\mu)(\gamma+\mu)}$ (10 marks)