MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND STATISTICS)

BACHELOR OF EDUCATION

SMA 409: ALGEBRAIC GEOMETRY

DATE: 6/5/2019 **TIME: 2:00 – 4:00 PM**

INSTRUCTIONS TO CANDIDATES

Answer question ONE and any other TWO questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) State the Fundamental theorem of algebra. (2 marks)

b) Briefly describe each of the following terms as used in algebraic geometry

i) Nullstellensatz for $\mathbb{C}[x]$. (2 marks)

ii) The Zariski Topology on $\mathbb{A}^n(F)$. (2 marks)

iii) Identify two examples of Affine algebraic sets. (2 marks)

c) Let F be a field

i) State the uniqueness factorization theorem in $F(x)$ (2 marks)

ii) Find the zeros of the polynomial $f(x) = x^2 + 2x + 4$ in $\mathbb{Z}_6[x]$ (3 marks)

iii) Factorize $f(x) = x^4 + 4$ in to linear factors in $\mathbb{Z}_5(x)$ (3 marks)

d) Define an Algebraic variety. (2 marks)

e) Let $S = \{3x - 4y - 12\} \subset \mathbb{R}[x, y]$ describe the algebraic variety $V(S)$ in $\mathbb{R}^2$. (3 marks)

f) Describe the algebraic variety V in $\mathbb{R}$ consisting of the common zeros of $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$ and $g(x) = x^3 - 3x^2 - 6x - 8$ (3 marks)

g) Show that if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is in $\mathbb{Z}[x]$ with $a_0 \neq 0$ and if $f(x)$ has a zero in in $\mathbb{Q}$, then it has a zero m in $\mathbb{Z}$ and m must divide a_0 . (3 marks)

h) Using the part (f) above discuss the irreducibility of the polynomial $f(x) = x^4 - 2x^2 + 8x + 1$ viewed in $\mathbb{Q}[x]$ over $\mathbb{Q}$. (3 marks)

<u>QUESTION two</u>

a) Briefly describe in your own words what we study in Algebraic geometry. **(2 marks)**

b) Briefly define the following

i) Evaluation homomorphism. **[2marks]**

ii) Hilbert Basis Theorem **[2marks]**

c) Compute the following evaluation homomorphisms:

i) $\emptyset_i(2x^3 - x^2 + 3x + 2)$ **[2marks]**

ii) $\emptyset_5(x^3 + 2)(4x^2 + 3)$ **[2marks]**

d) Perform a single step division algorithm reduction that changes the given basis to one having smaller maximum term order using order lex $z < y < x$.

i. $\langle xy + y^3,\ y^3 + z\ , x - y^4\ \rangle$ **[2 marks]**

ii. $\langle y^2z^3 + 3,\ y^3z^2 - 2z\ , y^2z^2 + 3\ \rangle$ **[2 marks]**

e) Show that the set of common zeros in F^n of the polynomials $f_1, f_2, \ldots f_n \in F[x]$,where F is a ring is the same as set of common zeros in F^n of all the polynomials in the entire ideal $I =< f_1, f_2, \ldots f_n >$. (6marks)

<u>QUESTION 3</u>

a) Give the definition of a Grobner basis. [2marks]

b) Identify some importances of Grobner basis in applications. **[2marks]**

c) Find a Grobner basis for the following ideal in $\mathbb{R}[x]$.

$\langle x^4 - 4x^3 + 5x^2 - 2x,\ \ x^3 - x^2 - 4x + 4\ ,\ \ x^3 - 3x + 2\ \rangle$ [4marks]

d) By division, reduce the basis $\{\ xy^2,\ y^2 - y\}$ for an ideal $I =< xy^2,\ y^2 - y >$ in $\mathbb{R}[x, y]$ to one with smaller term size assuming the order lex with $y < x$. [4marks]

e) Using appropriate method:

i) Describe the algebraic variety $V(< x+y-3z-8, 2x+y+z+5 >)$ in $\mathbb{R}^3$. **[4marks]**

ii) Describe the algebraic variety $V(< f(x,y), g(x,y) >)$ in $\mathbb{R}^2$ where:

$f(x,y) = x^2 y - 2,\ g(x,y) = \ xy^2 - y$ **[4marks]**

QUESTION 4

a) Give the definition of an irreducible polynomial $f(x) \in F[x]$. Give an example. (2marks) .

b) Prove that a non- zero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in a field F . (5marks)

c) Let $f(x) = x^4 + 5x^3 - 3x^2$ and $g(x) = 5x^2 - x + 2$ in $\mathbb{Z}_{11}(x)$.

i) Find the product $f(x)g(x)$. (3 marks)

ii) Use the division algorithm to determine the g.c.d of $f(x)$ and $g(x)$. (3 marks)

iii) Describe the kernel of $g(x)$. (3 marks)

d) Find a Grobner basis for the following ideal in $\mathbb{R}[x]$.

$\langle x^4 - 4x^3 + 5x^2 - 2x,\ \ x^3 - x^2 - 4x + 4\ ,\ \ x^3 - 3x + 2\ \rangle$ **[4marks]**

QUESTION 5

a) State and prove Hilbert Nullstellensatz **[7marks**

b) Show that if $f(x) = g(x)q(x) + r(x)$ then the common divisors in $F[x]$ of $f(x)$ and then the common divisors in $F[x]$ of $f(x)$ are the same as the common divisors in the $F[x]$ of $f(x)$ and $r(x)$ **[7 marks]**

c) Let F be a field. Show that if S is nonempty subset of F^n then $I(S) = \{f(x) \in F[x]: f(S) = 0\ \forall\ s \in S$ is an ideal of $F[x]$ **[6 marks]**