MACHAKOS UNIVERSITY

UNIVERSITY EXAMINATIONS 2022/2023

THIRD YEAR SECOND SEMESTER SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION (SCIENCE)

BACHELOR OF EDUCATION (ARTS)

SMA 331 DYNAMICS II

DATE: **TIME:**

Instructions

Answer question one (**compulsory**) and any other **two** questions

Question one (30 marks)

a) Let $\boldsymbol{F}(r) =- \boldsymbol{V}^{'}(r)$ for some function $\boldsymbol{V}$. Prove that the central force $\boldsymbol{F}(r) = \boldsymbol{F}(r)\hat{\boldsymbol{i}}$ is conservative with the potential function $\boldsymbol{V}^{'}(r)$. (3 marks)

b) A particle of mass m moving with a velocity $\boldsymbol{v}$ has an angular momentum $\boldsymbol{L} = m(\boldsymbol{r} \times \boldsymbol{v})$ at any position $\boldsymbol{r}$. Given that the angular momentum $\boldsymbol{L}$ points in the z-direction. Using polar coordinates, show that the magnitude of angular momentum is given by $L = mr^2$ (8 marks)

c) A small body of mass m slides on a rod which is the chord of a circular wheel. The wheel rotates about its centre O with a clockwise angular velocity $\omega = 2\, rad/sec$ and a clockwise angular acceleration $\dot{\omega} = 2.5\, rad/sec^2$. The body m has a constant velocity of $1.83\, m/s$ to the right, relative to the wheel. Given that $\rho = 0.22875\, m$ and $R = 0.4575\, m$, determine the absolute acceleration of the mass. (5 marks)

d) Given that the function $y(x)$ has a graph passing through two points x_1 and x_2. By minimizing the integral;

$I = \int_{x_1}^{x_2} \left[1 + y'(x)^2\right]^{1/2} dx,$

Show that $y(x)$ is a linear function defining the shortest distance between the two points. (7 marks)

e) Consider a single particle of mass m, moving subject to a force field. If the vector from a fixed origin to the particle at a time t is denoted by $\boldsymbol{r}$. Using the Newton's laws of motion, derive the Hamilton's principle in its most general form. (7 marks)

Question Two (20 marks)

A planet of mass m orbiting around a star of mass M located at the origin experiences a central force $\boldsymbol{F} = -\frac{GMm}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}$ and has an energy $E = \frac{1}{2}mv^2 - \frac{K}{r}$, where $K = GMm$;

a) Using Runge' vector $\boldsymbol{R} = \hat{\boldsymbol{r}} + \frac{\boldsymbol{L}}{\boldsymbol{K}}(\hat{\boldsymbol{k}} \times \boldsymbol{v})$, show that R remains constant in time. (3 marks)

b) Given that $\hat{\boldsymbol{k}} \times \boldsymbol{v} = \dot{r}\hat{\boldsymbol{\theta}} - r\dot{\boldsymbol{\theta}}\hat{\boldsymbol{r}}$;

i) Derive an expression for the eccentricity e of the orbit of the planet. (4 marks)

ii) Show that the radius of the planet from the origin is given by

$$r = \frac{L^2}{Km}\left(\frac{1}{1 - e\cos\theta}\right)$$ (4 marks)

iii) By letting $\beta = \frac{L^2}{Km}$, define;

I. Aphelion distance. (1 marks)

II. Perihelion distance. (1 marks)

c) Prove that the orbits of planets are ellipses with the sun at one focus and are centered at $(ea, 0)$ having the semi-major axis a and semi-minor axis $b = \sqrt{1 - e^2a}$. (4 marks)

d) Prove that the period T of a revolution of orbit is proportional to $a^{3/2}$. (3 marks)

Question Three (20 marks)

a) Considering a moving coordinate system; Show that the general expression for the acceleration of a point refereed to a coordinate system which itself is moving is given by;

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\rho}\boldsymbol{r} + 2\boldsymbol{\omega} \times \dot{\rho}\boldsymbol{r}$$ (10 marks)

b) Two concentric circles of radius r and R rotate about the sane fixed centre O. The constant angular velocity of the large disk measured in a fixed system is given by $\boldsymbol{\Omega}$. The constant angular velocity of the small disk relative to the large disk $\boldsymbol{\omega}$. Find the acceleration of a point A, on the circumference of the small disk and indicate the acceleration of coriolis.

(10 marks)

Question Four (20 marks)

a) Given that the function $y(x)$ has a graph passing through two points x_1 and x_2. By minimizing the integral;

$I = \int_{x_1}^{x_2} 2\pi y(x)\left[1 + y^{'}(x)^2\right]^{1/2} dx,$

Determine the minimal surface of revolution passing through the given points. (10 marks)

b) Obtain the Lagrange equation for a point mass m suspended by an inextensible string of length l. (10 marks)

Question Five (20 marks)

a) Consider a one-dimensional simple harmonic oscillator whose kinetic energy

$T = \frac{1}{2}m\dot{x}^2$ and potential $V = \frac{1}{2}Kx^2$. Using Hamilton-Jacobi theory, obtain

(i) the position $x(t)$ of the oscillator (10 marks)

(ii) Momentum $\rho(t)$ of the oscillator. (5 marks)

b) The centre of mass of a rigid body of mass $50\ kg$ has a velocity of magnitude $40\ m/s$. At a given instant the moment of momentum of the body about the centre of mass is $H_c = 1500\boldsymbol{i} + 1000\boldsymbol{j} + 1200\boldsymbol{k}\ lb\ ft\ sec$. The inertia integrals about the same coordinate axes are;

$$I = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 40 & -20 \\ 0 & -20 & 30 \end{bmatrix} kg\, m\, s^2$$

Find the angular velocity and kinetic energy of the body at a given instant. (5 marks)