MACHAKOS UNIVERSITY

University Examinations 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF EDUCATION (SPECIAL EDUCATION)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION (SCIENCE)

BACHELOR OF EDUCATION (ARTS)

SMA 331 DYNAMICS II

DATE: 14/4/2023 **TIME: 2:00 – 4:00 PM**

INSTRUCTIONS

Answer question one (**compulsory**) and any other **two** questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Given that $\hat{\boldsymbol{r}} = \nabla r$, where r is the magnitude of the position vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, for a particle in space. Prove that $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ (3 marks)

b) For a central force $\boldsymbol{F} = -V'\hat{\boldsymbol{r}},$ define torque and hence prove that the angular momentum $\boldsymbol{L}$ is conserved. (4 marks)

c) A particle of mass m moving with a velocity $\boldsymbol{v}$ has an angular momentum $\boldsymbol{L} = m(\boldsymbol{r} \times \boldsymbol{v})$ at any position $\boldsymbol{r}$. Given that the angular momentum $\boldsymbol{L}$ points in the z-direction. Using polar coordinates, show that the angular part is governed by $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$ (4 marks)

d) A small body of mass m slides on a rod which is the chord of a circular wheel. The wheel rotates about its centre O with a clockwise angular velocity $\omega = 8\, rad/sec$ and a clockwise angular acceleration $\dot{\omega} = 10\, rad/sec^2$. The body m has a constant velocity of $3.66\, m/s$ to the right, relative to the wheel. Given that $\rho = 0.915\, m$ and $R = 1.83\, m$, determine;

(i) The absolute acceleration of the mass. (3 marks)

(ii) Coriolis acceleration. (2 marks)

e) Given that the function $y(x)$ has a graph passing through two points $(0,1)$ and $(1,1)$. Determine the possible extremal for the integral $I = \int_0^1 1 + [y'(x)]^2 \, dx$, Show that $y(x)$ is a linear function defining the shortest distance between the two points. (4 marks)

f) A rigid body performs plane motion such that all points of the body move parallel to the plane. Considering three points A, B and C in a moving coordinate system such that C is considered to be the origin of the moving coordinate system (x, y, z) with reference to a fixed coordinate system and the velocities at the other two points A and B are known, show that the angular velocity of the body is given by; $\boldsymbol{\omega} = \frac{V_A}{\rho_A} = \frac{V_B}{\rho_B}$ Where V_A and V_B are the velocities of the body at A and B respectively and ρ_A and ρ_B are the displacements of the body from C to A and B respectively. (3 marks)

g) According to the Einstein's special theory of relativity, the mass m of a particle moving with a speed v relative to the observer is given by $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ where m_0 the rest is mass and c is the speed of light (3×10^8m/s). Determine the percentage increase in rest mass of the planet when moving at $40000\ km/h$. (3 marks)

h) Consider a single particle of mass m, moving subject to a force field. If the vector from a fixed origin to the particle at a time t is denoted by $\boldsymbol{r}$. Using the Newton's laws of motion, derive the Hamilton's principle in its most general form. (4 marks)

QUESTION TWO (20 MARKS)

a) A planet of mass m orbiting around a star of mass M located at the origin experiences a central force $\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$ and has an energy $E = \frac{1}{2}mv^2 - \frac{K}{r}$, where $K = GMm$;

(i) Using Runge' vector $\boldsymbol{R} = \hat{\boldsymbol{r}} + \frac{\boldsymbol{L}}{\boldsymbol{K}}(\hat{\boldsymbol{k}} \times \boldsymbol{v})$, show that R remains constant in time. (3 marks)

(ii) Show that $\hat{\boldsymbol{k}} \times \boldsymbol{v} = \dot{r}\hat{\boldsymbol{\theta}} - r\dot{\boldsymbol{\theta}}\hat{\boldsymbol{r}}$ and hence derive an expression for the eccentricity e of the orbit of the planet. (4 marks)

(iii) Show that the radius of the planet from the origin is given by $r = \frac{L^2}{Km}\left(\frac{1}{1-e\cos\theta}\right)$ (4 marks)

(iv) By letting $\beta = \frac{L^2}{Km}$, define;

I. Aphelion distance. (1 mark)

II. Perihelion distance. (1 mark)

b) Two concentric circles of radius r and R rotate about the same fixed centre O. The constant angular velocity of the large disk measured in a fixed system is given by $\boldsymbol{\Omega}$. The constant angular velocity of the small disk relative to the large disk $\boldsymbol{\omega}$. Find the acceleration of a point A, on the circumference of the small disk and indicate the acceleration of coriolis.

(7 marks)

QUESTION THREE (20 MARKS)

a) Considering a moving coordinate system; Show that the general expression for the acceleration of a point refereed to a coordinate system which itself is moving is given by;

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}\boldsymbol{r} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}\boldsymbol{r}$$ (10 marks)

b) Planets of mass m orbit around the sun and considered to be located at the origin such that the radius of any planet from the origin is given by $r = \frac{L^2}{Km}\left(\frac{1}{1-e\cos\theta}\right)$

(i) Prove that the orbits of such planets are ellipses with the sun at one focus and are centered at $(ea, 0)$ having the semi-major axis a and semi-minor axis $b = \sqrt{1-e^2a}$.

(5 marks)

(ii) Prove that the length of the semi-major axis b is proportional to $a^{1/2}$ and hence show that the period of a revolution of an orbit $T = 2\pi\sqrt{\frac{m}{k}}a^{3/2}$ (5 marks)

QUESTION FOUR (20 MARKS)

a) From the Einstein's special theory of relativity, the mass m of a particle moving with a speed v relative to the observer is given by $m = \frac{m_0}{\sqrt{1-\beta^2}}$, where$\beta = \frac{v}{c}$, m_0 the rest is mass and c is the speed of light $(3 \times 10^8\text{m/s})$.

(i) Show that the time rate of doing work is $m_0c^2\frac{d}{dt}(1-\beta^2)^{-1/2}$ (3 marks)

(ii) Deduce from (i) above that the kinetic energy is given by

$$T = (m - m_0)c^2 \approx m_0c^2\left[(1-\beta^2)^{-1/2} - 1\right]$$ (3 marks)

(iii) If v is much less that c show that $T = \frac{1}{2}mv^2$ approximately. (4 marks)

b) Given that the function $y(x)$ has a graph passing through two points x_1 and x_2. By minimizing the integral $I = \int_{x_1}^{x_2} 2\pi y(x)[1 + y'(x)^2]^{1/2}\, dx$, determine the minimal surface of revolution passing through the given points. (5 marks)

c) The centre of mass of a rigid body of mass 100 kg has a velocity of magnitude 40 m/s. At a given instant the moment of momentum of the body about the centre of mass is $H_c = 3000\boldsymbol{i} + 2000\boldsymbol{j} + 2400\boldsymbol{k}$ kgms The inertia integrals about the same coordinate axes are;

$$I = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 80 & -40 \\ 0 & -40 & 60 \end{bmatrix} kgms^2$$

Find the angular velocity and kinetic energy of the body at a given instant. (5 marks)

QUESTION FIVE (20 MARKS)

a) Consider a one-dimensional simple harmonic oscillator whose kinetic energy $T = \frac{1}{2}m\dot{x}^2$ and potential $V = \frac{1}{2}Kx^2$. Using Hamilton-Jacobi theory, obtain an expression for;

(i) the position $x(t)$ of the oscillator (10 marks)

(ii) Momentum $\rho(t)$ of the oscillator. (5 marks)

b) Obtain the Lagrange's equation for a point mass m suspended by an inextensible string of length l and displaced through a small angle θ. (5 marks)