



MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR
BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

BACHELOR OF SCIENCE IN MATHEMATICS

BACHELOR OF EDUCATION

SMA 407: MEASURE THEORY

DATE: 24/3/2021

TIME: 2.00-4.00 PM

INSTRUCTIONS

Answer ALL the questions in Section A and ANY TWO Questions in Section B

SECTION A

QUESTION ONE (30 MARKS) (COMPULSORY)

- a) Define a measure? (3 marks)
- b) Prove that if μ is a measure defined on a σ -algebra $\mathfrak{X}$. Then μ is monotonic, that is if $E \subset F$, then $\mu(E) \leq \mu(F)$. Furthermore if $\mu(E) < \infty$, then $\mu(F - E) = \mu(F) - \mu(E)$. (5 marks)
- c) Prove that $\mu^*({x}) = 0$ for all $x \in \mathbb{R}$ (4 marks)
- d) Show that if $\mu^*(E) = 0$, then E is L-measurable. (5 marks)
- e) Prove that the space $(\mathbb{R}, \mathcal{M}, \mu)$, where μ is the lebesgue measure is complete. (5 marks)
- f) Prove that if $f, g: \mathfrak{X} \rightarrow \mathbb{R}$ are two $\mathfrak{X}$ -measurable functions and c be a real number then the function cf is $\mathfrak{X}$ -measurable (5 marks)
- g) Prove that if φ and ρ are simple functions in $M^+(X, \mathfrak{X})$ and $c \geq 0$, then $\int c\varphi d\rho = c \int \varphi d\rho$ (3 marks)

SECTION B

QUESTION TWO (20 MARKS)

- a) Prove that if $f, g: \rightarrow \mathbb{R}$ are two $\mathfrak{X}$ –measurable functions and c be a real number then the function
- i) f^2
 - ii) $|f|$ are $\mathfrak{X}$ –measurable. (5 marks)
- b) Let (f_n) be a sequence of $\mathfrak{X}$ –measurable functions $f_n: X \rightarrow \mathbb{R}$, then the functions $f_1 \cup f_2 \cup f_3 \cup \dots \cup f_n$ and $f_1 \cap f_2 \cap f_3 \cap \dots \cap f_n$ is $\mathfrak{X}$ –measurable. (5 marks)

QUESTION THREE (20 MARKS)

- a) State and prove monotone convergence theorem. (10 marks)
- b) Let $(X, \mathfrak{X})$ be a measurable space. Then if $f, g \in M^+(X, \mathfrak{X})$, then
- i) $f + g \in M^+(X, \mathfrak{X})$ (6 marks)
 - ii) $\int cf du = c \int f du$ (4 marks)

QUESTION FOUR (20 MARKS)

- a) Prove that if f and g both belong to M^+ and $f \leq g$, then $\int f du \leq \int g du$ (6 marks)
- b) Prove that if $f \in M^+$ and if $E, F \in \mathfrak{X}$, with $E \subset F$ then $\int_E f du \leq \int_F f du$ (7 marks)
- c) State and prove Fatous lemma. (7 marks)

QUESTION FIVE (20 MARKS)

- a) Define σ –algebra (5 marks)
- b) Prove that if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$, $A, B \in \mathfrak{R}$ (5 marks)
- c) Prove that $\mu^*(\emptyset) = 0$ (3 marks)
- d) Prove that lebesgue outer measure μ^* is countably sub additive. (7 marks)