MACHAKOS UNIVERSITY

University Examinations for 2024/2025 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR DEGREE IN BACHELOR OF SCIENCE IN MATHEMATICS
BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE
BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING
BACHELOR OF EDUCATION (SCIENCE)

SMA 404: COMPLEX ANALYSIS II

DATE: 23RD JANUARY, 2025 **TIME: 2.00 PM – 4.00 PM**

INSTRUCTIONS:
*Answer question **ONE (Compulsory)** and **any other TWO** questions*

QUESTION ONE - (30 MARKS)

(a) Explain the meaning of the term *isolated singularity* of a function f(z). **(2 Marks)**

(b) Analyze the poles of the function $f(z) = \frac{z-\pi i}{(z+1)^2(z^2+\pi^2)}$ **(4 Marks)**

(c) Evaluate $\oint_C \frac{3z^2+1}{(z-1)(z+2)^2}dz$, where C is the circle $|z| = 3$. **(5 Marks)**

(d) Consider the transformation defined by $w = z^3 + 2z + 1$. Compute:

(i) the coefficient of magnification at $z_0 = \frac{2}{3} + i$, **(4 Marks)**

(ii) the angle of rotation at $z_1 = 1 - i$, **(4 Marks)**

(iii) the critical points of the transformation. **(2 Marks)**

(e) Consider the function $u(x, y) = x^3 - axy^2$

(i) Determine the constant a such that $u(x, y)$ is a harmonic function. **(4 Marks)**

(ii) Use the *harmonic conjugate theorem* to determine the harmonic conjugate of the $u(x, y)$. **(5 Marks)**

QUESTION TWO - (20 MARKS)

(a) Using examples, distinguish between a *Conformal transformation*, and an *Isogonal transformation*. **(4 Marks)**

(b) Determine whether $f(z) = e^{(z+i)}$ satisfy the Schwarz reflection principle. **(5 Marks)**

(c) Compute the residue of $f(z) = \frac{3z+9}{z^3(z+1)}$ at each of its poles. **(5 Marks)**

(d) Apply the Cauchy principal value technique to find the integral of $\frac{x}{(x^2+9)^2}$ over the entire real line. **(6 Marks)**

QUESTION THREE - (20 MARKS)

(a) Explain the meaning of the term *analytic continuation*. **(2 Marks)**

(b) Compute the Maclaurin series for the function

$$f(z) = \frac{e^z}{1-z}$$

(4 Marks)

(c) Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. **(4 Marks)**

(d) Prove that the series $g(z) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1} \left(z + \frac{1}{2}\right)^n$ is an analytic continuation of the function $f(z) = \left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} z^n$ and sketch the region where the two functions are analytic. **(7 Marks)**

(e) State without proof the Schwarz reflection principle. **(3 Marks)**

QUESTION FOUR - (20 MARKS)

(a) Explain the meaning of the following terms:

(i) *harmonic function* $h(x, y)$, **(2 Marks)**

(ii) *harmonic conjugate* of a function $h(x, y)$. **(2 Marks)**

(b) Evaluate the contour integral $\oint_C \frac{(z-1)}{z(z+1)(4z-15)} dz$ where C is given to be the circle $|z| = 2$. **(5 Marks)**

(c) Determine the Laurent series for the function

$$f(z) = \frac{1}{1+z}$$

which are valid where $2 < |z| < \infty$. **(5 Marks)**

(d) State without proof the *Schwarz-Christoffel transformation* theorem. **(2 Marks)**

(e) If $w = \infty$ is mapped onto a vertex z_n on the z-plane, prove that the term containing $(w - u_n)$ in the Schwarz-Christoffel transformation can be replaced by 1. **(4 Marks)**

QUESTION FIVE - (20 MARKS)

(a) Consider the transformation $w = z^2$.

(i) Determine the angle of rotation at $1 + i$. **(2 Marks)**

(ii) Compute the coefficient of magnification at $1 + i$. **(2 Marks)**

(b) Consider the transformation given by $w = z^2$.

(i) Determine the image of the triangular region on the z-plane bounded by the lines $x = 1, y = 1$ and $x + y = 1$. **(5 Marks)**

(ii) Verify conformality at the point $z = i$ by considering the intersection of the lines $y = 1$ and $x + y = 1$. **(4 Marks)**

(c) Prove that for closed polygons, the sum of the exponents

$$\frac{\alpha_1}{\pi} - 1, \frac{\alpha_2}{\pi} - 1, \cdots \frac{\alpha_n}{\pi} - 1$$

in the Schwarz-Christoffel transformation is -2. **(7 Marks)**