

MACHAKOS UNIVERSITY

University Supplementary Examinations

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

Examinations for the degree of

Bachelor of education (science & arts), Bachelor of Science (statistics, mathematics)

SMA 403: GALOIS THEORY

Date:

Time: 2 Hours

Instructions to Candidates

Answer question ONE and any other TWO questions

QUESTION 1: 30 MARKS (COMPULSORY)

- a) Let $f(x) = x^4 - 12x^2 + 35$ Over the field $\mathbb{Q}$.
- i. State the fundamental theorem of Galois theory. (3 marks)
 - ii. Find the zeros of the polynomial $f(x)$. (4 marks)
 - iii. Find the splitting field F of the polynomial $f(x)$. (3 marks)
 - iv. Find the index of F in $\mathbb{Q}$. (3 marks)
 - v. Construct the Galois group $G(F/\mathbb{Q})$. (4 marks)
 - vi. Identify a subgroup of S_n isomorphic to $G(F/\mathbb{Q})$ (2 mark)
- b) By using part (d) above construct lattice diagrams of the subgroups of Galois group and subfields of the splitting field. (5 marks)
- c) Give the definition of an automorphism of a field K . (2 marks)
- d) Let $\{\sigma_i: i \in N\}$ be a collection of automorphism of a field F , show that $F(\sigma_i) = \{x \in F: \sigma_i(x) = x \forall \sigma_i\}$ is a subfield of F . (4 marks)

QUESTION 2

- a) Determine the solvability by radicals and compute the Galois group of the polynomial $f(x) = x^3 + 5$. (5 marks)
- b) automorphism in $G(E/F)$ defines a permutation of the roots of $f(x)$ that lie in E . (6 marks)
- c) Consider the polynomial $f(x) = x^{11} - 1$ over Q . Find the zeros of this polynomial. (4 marks)
- d) Identify the splitting field of $f(x) = x^{11} - 1$ over Q . (3 marks)
- e) State the Galois group of $f(x) = x^{11} - 1$ over Q and comment on the order and its nature. (2 marks)

QUESTION 3

- a) Let F be a field. Give definitions of the following terms:
- Normal extension of a field F . (2 marks)
 - Cyclotomic extension of a field F . (2 marks)
- b) Consider the Galois group $G(Q(\sqrt{2}, \sqrt{3})/Q)$
- Identify the elements of this Galois group. (4 marks)
 - The field $Q(\sqrt{2}, \sqrt{3})$ may be considered as a vector space. Identify the basis elements of this vector space and state its dimension. (4 marks)
 - Use the fundamental theorem of Galois group to show by lattice diagrams that the subgroups are isomorphic to the subfields. (4 marks)
- c) Find the minimal polynomial in (b) above. (4 marks)

QUESTION 4

- a) Show that the polynomial $x^n - 1$ is solvable by radicals over $\mathbb{Q}$. (6 marks)
- b) Derive the cubic formula for finding the zeros of the polynomial:
 $f(x) = ax^3 + bx^2 + cx + d$. (10 marks)
- c) Find the discriminant of the cubic equation $ax^3 + bx^2 + cx + d = 0$. (4marks)

QUESTION 5

a) We know that the cyclotomic polynomial

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \dots + x + 1$$

is irreducible over $\mathbb{Q}$ for every prime p . Let w be a zero of $\Phi_p(x)$ and consider the field $\mathbb{Q}(w)$.

i. Show that $w, w^2, \dots, w^{p-1}$ are distinct zeros of $\Phi_p(x)$, and conclude that they are all zeros of $\Phi_p(x)$. (5 marks)

ii. Show that $G(\mathbb{Q}(w)/\mathbb{Q})$ is abelian of order $p - 1$. (5 marks)

b) Let F be a field of characteristic zero and let $f(x) \in F[x]$ be a separable polynomial of degree n . If E is the splitting field of $f(x)$, let $\alpha_1, \dots, \alpha_n$ be the roots of $f(x)$ in E .

Let $\Delta = \prod_{i \neq j} (\alpha_i - \alpha_j)$. We define the discriminant of $f(x)$ to be Δ^2 .

i. If $f(x) = ax^2 + bx + c$, show that $\Delta^2 = b^2 - 4ac$. (5 marks)

ii. $f(x) = x^3 + px + q$, show that $\Delta^2 = 4q^3 - 27p^2$. (5 marks)