



MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FIRST YEAR FIRST SEMESTER EXAMINATION FOR

MASTER OF SCIENCE (MATHEMATICAL MODELLING AND COMPUTATION)

SMA 804: MODELING WITH ORDINARY DIFFERENTIAL EQUATIONS

DATE: 24/5/2019

TIME: 2:00 – 5:00 PM

INSTRUCTIONS TO CANDIDATES

QUESTION ONE (COMPULSORY) (20MARKS)

- a) State any four steps of mathematical modeling (4 marks)
- b) The non-dimensional Lokta-Volterra model is given by
$$\begin{aligned}\dot{u} &= u(1-u) \\ \dot{v} &= \alpha v(u-1)\end{aligned}$$
 where α is a positive constant. Given that the phase trajectories for this model are $\alpha u + v - \ln(u^\alpha v) = C$ where C is a constant. Sketch the phase trajectories on the uv phase plane and briefly explain the dynamics (6 marks)
- c) In aid of a suitable flow diagram, design basic SEIR model equations to describe the transmission of malaria from mosquito population to human population. (6 marks)
- d) Using the Liapunov test function $V(x, y) = x^2 + 2y^2 = a$, $a > 0$, show that the zero solution of the system $\dot{x} = -x - 2y^2$, $\dot{y} = xy - y^3$ is uniformly and asymptotically stable. (4 marks)

QUESTION TWO (20 MARKS)

- a) The spread of common cold/flu in a given confined population can be modeled using the SIR model:

$$\dot{S} = -\beta SI, \quad \beta > 0, S(0) > 0$$

$$\dot{I} = \beta SI - \sigma I, \quad \sigma > 0, I(0) > 0$$

$$\dot{R} = \sigma I, \quad R(0) = 0$$

- Sketch the flow diagram used to deduce the above model equations. (4 marks)
 - State any three assumptions used to develop the above SIR model that describes the spread of the common cold/flu. (3 marks)
 - Realistically, the population size changes and common cold is known to recur i.e. a recovered individual can get the disease once again. Improve the given model equations by accounting for this realistic phenomenon. (3 marks)
- b) The Lotka-Volterra model for two-species ecosystem is given by: $\begin{aligned} \dot{x} &= ax - cxy \\ \dot{y} &= -by + dxy \end{aligned}$ where a, b, c, d are positive constants.
- Determine the equilibrium points of the model (4 marks)
 - Determine the phase paths of this system of equations. (6 marks)

QUESTION THREE (20 MARKS)

- a) In an electrical circuit the current $I(t)$ is modeled by $L \frac{dI}{dt} + RI = E(t)$. Determine the current in a circuit where the resistance is 12Ω , the inductance is $4H$, a battery gives a constant voltage of $60V$, and the switch is turned on when $t = 0$. (7 marks)
- b) Let $\vec{x}^*(t)$ be a given solution vector of the non-autonomous system $\frac{d\vec{x}(t)}{dt} = \vec{X}(\vec{x}, t)$. Define Liapunov stability of the solution $\vec{x}^*(t)$ to such a system (3 marks)
- c) Given the dynamical system: $\begin{aligned} \dot{x} &= y(1 - x^2) \\ \dot{y} &= -x(1 - y^2) \end{aligned}$
- Determine the equilibrium points (5 marks)
 - Show that the phase paths are given by the equation $(1 - x^2)(1 - y^2) = C$, where C is a constant. (5 marks)

QUESTION FOUR (20 MARKS)

- a) Vibrations of a two-storey shear building can be modeled by the system of equations:
- $$\begin{aligned} m\ddot{x}_1 + 2kx_1 - kx_2 &= F_1(t) \\ m\ddot{x}_2 - kx_1 + kx_2 &= F_2(t) \end{aligned}$$
- Sketch the two-storey building indicating where the $m, k, x_1, x_2, F_1(t), F_2(t)$ lies in the diagram. (7 marks)
- b) Consider the predator, $P(t)$ -prey, $N(t)$ model $\begin{aligned} \dot{N} &= N[a - bP] \\ \dot{P} &= P[cN - d] \end{aligned}$ where a, b, c, d are positive constants. Improve this model such that each of the species $P(t)$ and $N(t)$ have logistic growth in the absence of the other and state the meaning of each constant in the improved model. (8 marks)
- c) One of the equations that is used to model vehicular traffic flow on a highway can be expressed in discrete form as $\rho(i, k+1)\Delta x - \rho(i, k)\Delta x = q(i-1, k)\Delta t - q(i, k)\Delta t$:
- State the assumption and principle used to arrive at the above equation. (3 marks)
 - Give the two instances that contribute to the change in traffic density at the time instance t_{k+1} (2 marks)

QUESTION FIVE (20 MARKS)

The SEIR model consists of the following models equations:

$$\begin{aligned} \dot{S} &= -\beta SI + \lambda - \mu S \\ \dot{E} &= \beta SI - (\mu + k)E \\ \dot{I} &= kE - (\gamma + \mu)I \\ \dot{R} &= \gamma I - \mu R \end{aligned}$$

- Sketch the flow diagram of the above model equations. (5 marks)
- State what each of the following parameters $\beta, \lambda, \mu, k, \gamma$ represents. (5 marks)
- Obtain the matrices F and V^{-1} for this model, hence show that the reproduction number is given by $R_0 = \frac{k\beta\lambda}{\mu(k + \mu)(\gamma + \mu)}$ (10 marks)