MACHAKOS UNIVERSITY

University Examinations 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION (SCIENCE)

SMA 303: RING THEORY

DATE: 14/4/2023 **TIME: 8:30 – 10:30 AM**

INSTRUCTIONS: Answer Question One and any other two questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Explain the meaning of the following terms:

(i) an *zero divisor* in a ring R, (3 marks)

(ii) a *invertible element* in a ring R. (3 marks)

b) Suppose that R is a ring and let $a, b, c, d \in R$ be elements of R.

(i) State the distibutivity axioms in a ring R. (4 marks)

(ii) Using part (i) above or otherwise, prove that

$$(a+b)(c+d) = ac + ad + bc + bd.$$

(iii) Justify each step in your solution. (4 marks)

c) Let $\mathbb{Q}$ denote the ring of rational numbers and consider the subset

$$S = \left\{\frac{a}{3} \mid a \in \mathbb{Z}\right\}.$$

i.e. S contain all the rational numbers that can be written in the form $\frac{a}{3}$ for some $a \in \mathbb{Z}$. Determine whether S a subring of $\mathbb{Q}$. (5 marks)

d) Let $(\mathbb{Z}_{24},\oplus,\otimes)$ denote the ring of integers modulo 24 under addition and multiplication modulo 24. Let I be the *principal ideal* of $\mathbb{Z}_{24}$ generated by 4.

(i) Determine all elements of I. (4 marks)

(ii) Construct all the elements of the *quotient ring* $\mathbb{Z}_{24}/I$. (4 marks)

e) Suppose that A and B are rings and that $f: A \to B$ be a surjective ring homomorphism. Prove if B is commutative then A is also a commutative ring. (4 marks)

QUESTION TWO (20 MARKS)

a) Distinguish between a *division ring* and a *field*. (5 marks)

b) Suppose that R is a commutative ring and that $a \in R$ is an invertible element. Prove that if $ab = 1$ and that $ac = 1$ then $b = c$. (5 marks)

c) Let $\mathbb{Z}[i] = \{x + iy \mid x, y \in \mathbb{Z}\} \subset \mathbb{C}$ be the set of Gaussian integers. Determine all the units in $\mathbb{Z}[i]$. (6 marks)

d) Suppose that $\varphi: A \to B$ is a surjective ring homomorphism. Prove that

(i) if $I \subset A$ is a prime ideal then the image $\varphi(I)$ is a prime ideal of B, (3 marks)

(ii) if $J \subset B$ is a prime ideal then $\varphi^{-1}(J) = \{a \in A \mid \varphi(a) \in J\}$ is also a prime ideal. (3 marks)

QUESTION THREE (20 MARKS)

a) Explain the meaning of the following terms.

(i) an *ideal* I in a ring R, (2 marks)

(ii) a *quotient* of a ring R by an ideal I. (2 marks)

b) Let $M_2(\mathbb{Z})$ be the ring of all 2×2 matrices with integer entries. Consider the sub $L \subset M_2(\mathbb{Z})$set given by

$$L = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

i.e. L is the subset containing lower triangular matrices. Determine whether T is a subring of $M_2(\mathbb{Z})$. (5 marks)

c) Suppose that A and B are rings. Prove that if $\varphi: A \to B$ is a ring homomorphism then

(i) $\varphi(A)$ is a subring of B, (4 marks)

(ii) $Ker\ \varphi$ is an ideal of A, (4 marks)

(iii) if A is a commutative ring then $\varphi(A)$ is commutative. (3 marks)

QUESTION FOUR (20 MARKS)

a) Explain the meaning of the following terms (4 marks)

(i) a *prime ideal* in a ring R,

(ii) a *maximal ideal* in a ring R.

b) Suppose that S_1 and S_2 are subrings of a ring R. Prove that $U \cap V$ is also a subring of R. (4 marks)

c) Let n be a fixed positive integer. Use an example to distinguish between a ring of characteristic n and a ring of characteristic 0. (4 marks)

d) Suppose that $\varphi: A \to B$ is a ring homomorphism and let $a \in A$.

(i) Show that $\varphi(a^n) = \varphi(a)^n$. (4 marks)

(ii) Show that if a is a unit in A then $\varphi(a)$ is a unit in B and $\varphi(a)^{-1} = \varphi(a^{-1})$. (4 marks)

QUESTION FIVE (20 MARKS)

a) Explain the meaning of the following terms

i) a ring *homomorphism* $\varphi: A \to B$, (2 marks)

ii) a *kernel* of a ring homomorphism $\varphi: A \to B$. (2 marks)

b) Let R be a commutative ring.

(i) Prove that $\{0\}$ and R are the only ideals of R if and only if R is a field. (3 marks)

(ii) Using (i) above or otherwise, prove that the ideal $\{0\}$ is maximal if and only if R is a field. (3 marks)

c) Let $\mathbb{Q}[\sqrt{2}]$ denote the set $\{x + y\sqrt{2}: x, y \in \mathbb{Q}\}$. Define the binary operations $+$ and $\cdot$ on $\mathbb{Q}[\sqrt{2}]$ as follows

$$\begin{aligned}(x + y\sqrt{2}) + (z + w\sqrt{2}) &= (x + y) + (y + w)\sqrt{2} \\ (x + y\sqrt{2}) \cdot (z + w\sqrt{2}) &= (xz + 2yw) + (xw + yz)\sqrt{2}\end{aligned}$$

(i) Prove that $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$. (4 marks)

(ii) Determine whether $\mathbb{Q}[\sqrt{2}]$ is an integral domain. (6 marks)