MACHAKOS UNIVERSITY

University Examinations 2019/2020 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING

BACHELOR OF SCIENCE IN MATHEMATICS

BACHELOR OF EDUCATION

SMA 402: FIELD THEORY

DATE: 18/01/2021 **TIME: 8.30-10.30 AM**

INSTRUCTIONS TO CANDIDATES

Answer question ONE and any other TWO questions

QUESTION ONE 30 MARKS (COMPULSORY)

a) Using relevant examples differentiate between a finite Field and a Field of characteristic zero. Give Two examples of each. (3 marks)

b) Let $x = \sqrt{3+\sqrt{5}}$. Find the minimal polynomial of x over $\mathbb{Q}$ (3 marks)

c) Find the splitting field of the polynomial $P(x) = x^4 - 10x + 21 \in Q[x]$.
Hence identify the basis and dimension of the basis. (4 marks)

d) Show that $Q(\sqrt{3}) = \{a + b\sqrt{3} : a , b \in Q\}$ is a Field. (4 marks)

e) Consider the polynomial in $p(x) = x^2 + x + 1$ in$Z_2\,[x]$

   i. Describe what is meant by an irreducible polynomial. (2 marks)

   ii. Verify that the polynomial $p(x)$ is irreducible in Z_2 (1 mark)

   iii. Let $Z_2\,[\alpha]$ be the extension field of Z_2 where α is algebraic over Z_2 . In terms of α list and simplify all the elements of $Z_2\,[\alpha]$. Construct addition and multiplication tables of $Z_2\,[\alpha]$ then verify that $Z_2\,[\alpha]$ is a field. (9 marks)

iv. Find **irr**(α, Q) and **deg**(α, Q) for the given algebraic number $\alpha \in C$ over Q where $\alpha = \sqrt{5} + \boldsymbol{i}$ (3 marks)

QUESTION TWO (20 MARKS)

a) Give definitions of the following terms giving examples in each:

i. An algebraic element α of an extension field E of a field F. (2 marks)

ii. A transcendental element α of an extension field E of a field F. (2 marks)

b) Let $f(x) = x^4 - 8x^2 + 15$ Over the field Q.

i. Find the zeros of the polynomial $f(x)$. (4 marks)

ii. Find the splitting field F of the polynomial $f(x)$. (2 marks)

iii. Exhibit a tower of fields in (ii) above (2 marks)

iv. Find the index of F in Q. (2 marks)

c) i. Describe what is meant by intersection of subfields: Give an example. (3 marks)

ii. Prove that $x^2 - 3$ is irreducible over $Q(\sqrt[3]{2})$ (3 marks)

QUESTION THREE (20 MARKS)

a) (i) Show that the polynomial $x^2 + 1$ is irreducible in $Z_3[x]$ (3 marks)

(ii) Let β be a zero of the polynomials $x^2 + 1$ in the extension field $Z_3[\beta]$.

List down the nine elements of $Z_3[\beta]$. Give the addition and multiplication tables. Justify that $Z_3[\beta]$ is the finite field $GF(3^2)$ (8 marks)

b) Show that a field F is algebraically closed if and only if every non constant polynomial in $F[x]$ factors in $F[x]$ into linear factors. (5 marks)

c) Find the degree and the basis for the field extension $Q(\sqrt[3]{2}\ ,\ \sqrt{3}\,)$ over Q . Hence construct the tower of fields clearly indicating all the subfields. (4 marks) .

QUESTION FOUR (20 MARKS)

a) Find a basis for the extension field $Q(\sqrt{5} + \boldsymbol{i}\,)$ over Q (3 marks)

b) Show that if E is a finite extension field of a field F and K is a finite extension field of E then K is a finite extension of F and $[K\colon F] = [K\colon E][E\colon F]$ (7 marks)

c) Is the polynomial $p(x) = x^3 + 2x + 3$ in $Z_5[x]$ irreducible? Why? Express it as a product of irreducible polynomials in $Z_5[x]$ (5 marks)

d) Find all the generators of the cyclic multiplicative group of units of the field Z_{11} (5 marks)

QUESTION FIVE (20 MARKS)

a) i) Show that it is impossible by straight edge and compass alone to trisect an angle of 60^0 (4 marks)

ii) Show that it is impossible by straight edge and compass alone to duplicate the cube. (4 marks)

b) Write $F_3 = \{0,1,2\}$, the finite field of order 3 with arithmetic performed modulo 3. Let $\alpha = \sqrt{2}$.

i) Is $2 + \alpha$ a square in $F_3(\alpha)$. (3 marks)

ii) Let $\beta = \sqrt{2 + \sqrt{2}}$, calculate the degree of the extension $F_3(\beta): F_3(\alpha)$ (3 marks)

iii) Calculate the degree of the extension $F_3(\beta): F_3$ (3 marks)

iv) Justify that the set $\{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in Q\}$ is a field. (3marks)