MACHAKOS UNIVERSITY

University Examinations for 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FIRST YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (ACTUARIAL SCIENCE)

SAC 102: MATHEMATICAL MODELLING

DATE: 12/4/2023 **TIME: 2:00 – 4:00 PM**

INSTRUCTIONS

Answer Question ONE and ANY OTHER TWO

QUESTION ONE (30 MARKS)

a) Explain briefly two objectives of mathematical modelling giving an example. (2 marks)

b) Explain briefly two main categories of mathematical models. (4 marks)

c) A disease spreads in such a way that 100 individuals become infected during any year and 25% of those infected at the beginning of a year die before the end of the year. Write down a difference equation for I_n, the number of people infected at the end of n years. What happens in the long term? (4 marks)

d) Explain briefly two sources of errors in the modelling process. (4 marks)

e) A study was made by a retail merchant to determine the relation between weekly advertising expenditure and sales. The following data were recorded ($):

Advertising costs	40	20	25	20	30	50	40	20	50	40	25	50
Sales	385	400	395	365	475	440	490	420	560	525	480	510

Fit a straight line to predict weekly sales from advertising expenditures. (6 marks)

f) A logistic population model is used to estimate the population p(t), t years after the initial observation of the population of the town of Machakos.
Population p(t) is given by:

$$P(t)=\frac{50000}{2+3e^{-0.04t}}$$

i) Write down the initial and ultimate population sizes. (2 marks)

ii) Calculate, correct to the nearest whole number whole number the rate of growth of the population 10years after the initial observation. (3 marks)

g) Solve the initial value problem

$$\begin{cases} \frac{dx}{dt} = \alpha x \\ x(0) = x_0 \end{cases}$$

where $\alpha = constant.$ (5 marks)

QUESTION TWO (20 MARKS)

a) Determine the solution of the following difference equation

$$a_{n+1} = 5a_n, a_0 = 10$$

(2 marks)

b) A sewage treatment plant processes raw sewage to produce usable fertilizer and clean water by removing all other contaminants. The process is such that each hour 12% of remaining contaminants in a processing tank are removed.

i. What percentage of the sewage would remain after 1 day? (4 marks)

ii. How long would it take to lower the amount of sewage by half? (3 marks)

iii. How long until the level of sewage is down to 10% of the original level? (2 marks)

c) Your grandparents have an annuity. The value of the annuity increases each month as 1% interest on the previous months balance is deposited. They withdraw Ksh 1,000 each month for living expenses. Currently, they have Ksh 50,000 in the annuity.

i. Model the annuity with a dynamical system. (2 marks)

ii. Determine the equilibrium value. Comment on its stability. (3 marks)

iii. Build a numerical solution to determine when the annuity is depleted. (4 marks)

QUESTION THREE (20 MARKS)

a) Explain briefly one reason why the linear model for population growth is unrealistic. (3 marks)

b) A car rental company has distributorships in Nairobi and Mombasa. The company specializes in catering to travel agents who want to arrange tourist activities in both cities. Consequently, a traveller will rent a car in one city and drop the car off in the second city. Travellers may begin their itinerary in either city. The company wants to determine how much to charge for this drop-off convenience. Historical records reveal that 60% of the cars rented in Nairobi are returned to Nairobi, whereas 40% end up in Mombasa. Of the cars rented in the Mombasa office, 70% are returned to Mombasa, whereas 30% end up in Nairobi. Because cars are dropped off in both cities, will a sufficient number of cars end up in each city to satisfy the demand for cars in that city? If, not, how many cars must the company transport from Nairobi to Mombasa or from Mombasa to Nairobi?

i) Sketch a diagram to represent the situation. (1 mark)

ii) Let n denote the number of business days, $N_n =$ the number of cars in Nairobi at the end of day n and $M_n =$ the number of cars in Mombasa at the end of day n. Develop a model for the system. (2 marks)

iii) Determine the equilibrium values for the system. (3 marks)

iv) Explore the sensitivity of the equilibrium values for the following initial conditions

	Nairobi	**Mombasa**
Case 1	7,000	0
Case 2	2,000	5,000

(4 marks)

c) The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled and after 3yrs the population is 20,000. Determine the number of people initially in the country. (7 marks)

QUESTION FOUR (20 MARKS)

a) Consider the data in the table below.

x	17	19	20	22	23	25	28	31	32	33	36	37	39	42
y	19	25	32	51	57	71	113	140	153	187	192	205	250	260

i. Use the least squares criterion to derive the estimate of A for the line $y = Ax^n$, n fixed. (4 marks)

ii. Fit the line $y = Ax^2$ to the data in the table. (4 marks)

iii. Predict the value of y when $x = 40$. (1 mark)

b) An investor invests an initial amount P_0. He then adds a regular amount at the end of each year.

i. Obtain an expression for how much the sum of money grows to assuming an interest rate of r% p.a. (3 marks)

ii. What will be the amount assuming $P_0 = £10{,}000$, $r = 12\%$ and the regular amount added is £500? (2 marks)

c) Consider the logistic growth model $\frac{dp}{dt} = \alpha p(1 - \frac{p}{N})$. Determine its equilibrium points and by aid of a phase portrait diagram analysis comment on their stability. (6 marks)

QUESTION FIVE (20 MARKS)

a) Integrate $\frac{dy}{dx} = 3x^2e^{-y}$ by separating the variables first. (4 marks)

b) Consider the spreading of a highly communicable disease on an isolated island with population size N. A portion of the population travels abroad and returns to the island infected with the disease. You would like to predict the number of people X who will have been infected by some time *t*. Consider the following model, where $k>0$ is constant:

$$\frac{dX}{dt} = kX(N - X)$$

i) List two major assumptions implicit in the preceding model. (2 marks)

ii) Solve the model for X as a function of t. (6 marks)

c) Consider the differential equation $\frac{dy}{dt} = 2 - 3y + y^2$.

i) Determine its equilibrium solutions.

ii) Analyse the solutions obtained in (i) above for stability by sketching the general solution and the phase diagram. (8 marks)