MACHAKOS UNIVERSITY

University Supplementary Examinations

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

Examinations for the degree of

Bachelor of education (science & arts), Bachelor of Science (statistics, mathematics)

SMA 403: GALOIS THEORY

Date: …………………….. **Time: 2 Hours**

Instructions to Candidates

Answer question ONE and any other TWO questions

QUESTION 1: 30 MARKS (COMPULSORY)

a) Let $f(x) = x^4 - 12x^2 + 35$ Over the field $\mathbb{Q}$.

i. State the fundamental theorem of Galois theory. (3 marks)

ii. Find the zeros of the polynomial $f(x)$. (4 marks)

iii. Find the splitting field F of the polynomial $f(x)$. (3 marks)

iv. Find the index of F in $\mathbb{Q}$. (3 marks)

v. Construct the Galois group $G(F/\mathbb{Q})$. (4 marks)

vi. Identify a subgroup of S_n isomorphic to $G(F/\mathbb{Q})$ (2 mark)

b) By using part (d) above construct lattice diagrams of the subgroups of Galois group and subfields of the splitting field. (5 marks)

c) Give the definition of an automorphism of a field K. (2 marks)

d) Let $\{\sigma_i : i \epsilon N\}$ be a collection of automorphism of a field F, show that

$F(\sigma_i) = \{x \epsilon F : \sigma_i(x) = x \ \forall \ \sigma_i\}$ is a subfield of F. (4 marks)

QUESTION 2

a) Determine the solvability by radicals and compute the Galois group of the polynomial $f(x) = x^3 + 5$. (5 marks)

b) automorphism in $G(E/F)$ defines a permutation of the roots of $f(x)$ that lie in E. (6 marks)

c) Consider the polynomial $f(x) = x^{11} - 1$ over Q. Find the zeros of this polynomial. (4 marks)

d) Identify the splitting field of $f(x) = x^{11} - 1$ over Q. (3 marks)

e) State the Galois group of $f(x) = x^{11} - 1$ over Q and comment on the order and its nature. (2 marks)

QUESTION 3

a) Let F be a field. Give definitions of the following terms:

   i. Normal extension of a field F. (2 marks)

   ii. Cyclotomic extension of a field F. (2 marks)

b) Consider the Galois group $G(Q(\sqrt{2}, \sqrt{3})/Q)$

   i. Identify the elements of this Galois group. (4 marks)

   ii. The field $Q(\sqrt{2}, \sqrt{3})$ may be considered as a vector space. Identify the basis elements of this vector space and state its dimension. (4 marks)

   iii. Use the fundamental theorem of Galois group to show by lattice diagrams that the subgroups are isomorphic to the subfields. (4 marks)

c) Find the minimal polynomial in (b) above. (4 marks)

QUESTION 4

a) Show that the polynomial $x^n - 1$ is solvable by radicals over $\mathbb{Q}$. (6 marks)

b) Derive the cubic formula for finding the zeros of the polynomial:
   $f(x) = ax^3 + bx^2 + cx + d$. (10 marks)

c) Find the discriminant of the cubic equation $ax^3 + bx^2 + cx + d = 0$. (4marks)

QUESTION 5

a) We know that the cyclotomic polynomial

$$\emptyset p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \ ... + x + 1$$

Is irreducible over Q for every prime p. Let w be a zero of $\emptyset p(x)$ and consider the field $Q(w)$.

i. Show that $w, w^2, \ldots\ w^{p-1}$ are distinct zeros of $\emptyset p(x)$, and conclude that they are all zeros of $\emptyset p(x)$. (5 marks)

ii. Show that $G(Q(w)/Q)$ is abelian of order $p - 1$. (5 marks)

b) Let F be a field of characteristic zero and let $f(x) \in f[x]$ be a separable polynomial of degree n. If E is the splitting field of $f(x)$, let $\propto_1, \ldots\ \propto_n$ be the roots of $f(x)$ in E.

Let $\Delta = \pi_{i \neq j}(\propto_i - \propto_j)$. We define the disciminant of $f(x)$ to be Δ^2.

i. If $f(x) = ax^2 + \ bx + c$, show that $\Delta^2 = b^2 - 4ac$. (5 marks)

ii. $f(x) = x^3 + \ px + q$, show that $\Delta^2 = 4q^3 - 27q^2$. (5 marks)