MACHAKOS UNIVERSITY

University Examinations 2019/2020 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING

SMA 460: STOCHASTIC PROCESSES

DATE: 22/1/2021 **TIME: 2.00-4.00 PM**

INSTRUCTIONS

Answer ALL the questions in Section A and ANY TWO Questions in Section B

SECTION A

QUESTION ONE (30 MARKS) (COMPULSORY)

a) Define the following terms as used in stochastic processes;

i) Generating function (1 mark)

ii) Homogeneous markov chain. (1 mark)

iii) A stationary random process. (1 mark)

b) Let X have the distribution of geometric form given by

$$P(X=k)=\begin{cases} q^{k-2} & k=2,3\dots. \\ 0 & elsewhere \end{cases}$$

Find:

i. The probability generating function of X. (3 marks)

ii. Mean. (2 marks)

iii. Variance. (3 marks)

c) In the Dark Ages, Harvard, Dartmouth, and Yale Universities admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale. Find the probability that the grandson of a man from Harvard went to Harvard. Find the probability that the grandson of a man from Harvard went to Harvard. (4 marks)

d) Consider the differential-difference equations for the Poisson process i.e.

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \quad n \geq 1$$

and

$$P_0'(t) = -\lambda_0 P_0(t) \qquad n = 0$$

i. Obtain the probability distribution of this process by solving the difference differential equations under the initial conditions;

$$P_n(0) = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$ (8 marks)

ii. Compute the mean of this process. (4 marks)

e) Suppose that customers arrive at a Poisson rate of 1 per every 12 minutes and that the service time is exponential at a rate of 1 service per 8 minutes. Calculate:

i. Average number of customers in the system. (2 marks)

ii. Average amount of time the customers spends waiting in the system. (2 marks)

SECTION B

QUESTION TWO (20 MARKS)

a) Define the following terms:

i. Absorbing state. (1 mark)

ii. Irreducible Markov chain. (1 mark)

iii. Period of a state of Markov chain. (1 mark)

b) Find the generating function of the Fibonacci numbers {an} defined by

$a_n = a_{n-1} + a_{n-2}, \quad n \geq 2$

where $a_0 = 0$ and $a_1 = 1$. (3 marks)

c) Define the term Markov chain and determine whether the following chain is irreducible:

$$\begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (4 marks)

d) Classify the states of the following Markov chain. (10 marks)

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}$$

QUESTION THREE (20 MARKS)

a) Distinguish between a discrete time stochastic process and continuous time stochastic process. (2 marks)

b) Suppose N has a poisson distribution with mean λ and $X_i, i - 1, 2, \ldots, N$ are iid with p.g.f $P(s)$. Find the p.g.f. of $S_N = X_1 + X_2, + \cdots + X_N$. (5 marks)

c) The number of particles emitted by a radioactive source is Poisson distributed. The source emits particles at a rate of 6 per minute. Each emitted particle has a probability of 0.7 of being counted. Find the probability that 11 particles are counted in 4 minutes. (4 marks)

d) In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average of 36 minutes; calculate

i. Expected queue size (line length). (2 marks)

ii. Probability that the queue size exceeds 10. (2 marks)

iii. If the input of trains increases to an average of 33 per day, what will be the change in (a) and (b). (5 marks)

QUESTION FOUR (20 MARKS)

a) Distinguish between a weakly stationary process and strictly stationary process. (4 marks)

b) Let X be a random variable with p.g.f $P(s)$. Find the p.g.f of the random variable $Y = mX + n$, where m, n are integers and $m \neq 0$. (4 marks)

c) Consider the differential-difference equations for the Polya process i.e.

$$P_n'(t) = -\lambda \left(\frac{1+an}{1+\lambda at}\right) P_n(t) + \lambda \left(\frac{1+a(n-1)}{1+\lambda at}\right) P_{n-1}(t) \qquad n \geq 1$$

and

$$P_0'(t) = -\left(\frac{\lambda}{1+\lambda at}\right) P_0(t) \qquad n = 0$$

i. Obtain the probability distribution of this process by solving the difference differential equations under the initial conditions;

$$P_n(0) = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$ (9 marks)

ii. Compute the mean of this process. (3 marks)

QUESTION FIVE (20 MARKS)

a) Distinguish between a recurrent and a non-recurrent state in a Markov chain. (2 marks)

b) A message transmission system is found to be Markovian with the transition probability of current message to the next message as given by the matrix

$$P = \begin{bmatrix} 0.20 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

The initial probabilities of the states are: $p_1(0) = 0.4,\ p_{2(0)} = 0.3,\ p_3(0) = 0.3$. Find the probabilities of the next message. (7 marks)

c) Consider and M/M/1 queuing system with an interval rate $\lambda = 0.4$ and service rate $\mu = 0.5$.

i) Compute the system load and tell if the system stable or not? (2 marks)

ii) Compute the average number of customers in the system? (2 marks)

d) A person owning a tuktuk has the option to switch over to tuktuk, bike of a car next time with a probability of $(0.3, 0.5, 0.2)$.if the transition probability matrix is;

$$P = \begin{bmatrix} 0.4 & 0.3 & c \\ b & 0.5 & 0.3 \\ 0.25 & a & 0.5 \end{bmatrix}$$

i) Find the value of $a,\ b$ and c. (3 marks)

ii) What are the probabilities of the vehicles related to his fourth purchase. (4 marks)