MACHAKOS UNIVERSITY

UNIVERSITY SUPPLEMENTARY EXAMINATIONS

FOURTH YEAR SECOND SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE IN (STATISTICS AND PROGRAMMING , MATHEMATICS AND COMPUTER SCIENCE) BACHELOR OF ARTS IN MATHEMATICS AND ECONOMICS, BACHELOR OF EDUCATION (SCIENCE, ARTS)

SMA 465: MEASURE AND PROBABILITY.

DATE : TIME: 2 HOURS

INSTRUCTIONS: Attempt Question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

a. Define the set Ω, outlining any three basic operations on subsets of Ω. (4 marks)

b. Prove the assertions:

  i. $\underline{\lim} A_n \subseteq \overline{\lim} A_n$. (1 mark)

  ii. $\left(\underline{\lim} A_n\right)^c = \overline{\lim} A_n{}^c$. (3 marks)

c. Show that the class I of all intervals of $\mathbb{R}$ is a semi-algebra. (6 marks)

d. Explain any three examples of measure. (3 marks)

e. Let A be $\sigma - a\lg ebra$ of subsets of Ω. Let m be an application from A to $\mathbb{R}_+$. Suppose that we have: For all sequences $(A_n)_{n\geq 0}$ of pairwise disjoint elements of A, the σ-additivity formula holds $m\left(\sum_{n\geq 0} A_n\right) = \sum_{n\geq 0} m(A_n)$. Show that the two assertions are equivalent:

- $m(\phi) = 0$.
- m is a proper application constantly equal to $+\infty$, that is $\exists A \in \mathrm{A}, m(A) < +\infty$.

Show why the results you obtain still holds if m is only finitely additive on an algebra. (3 marks)

f. Let Ω be a nonempty set and put $\mathrm{A} = \{A \subset \Omega, A \textit{ is countable or } A^c \textit{ is countable}\}$.

i. Show that A is a $\sigma - a\lg ebra$. (4 marks)

ii. Let $\Omega = [0,1]$. Show that $\left[0, \frac{1}{2}\right]$ is not measurable in the measurable space (Ω, A). (2 marks)

g. Consider the Lebesque measure λ on $\mathbb{R}$.

i. Show that for any $x \in \mathbb{R}$, $\lambda(\{x\}) = 0$. (3 marks)

ii. Show that for countable A subset of $\mathbb{R}$, $\lambda(\{A\}) = 0$. (1 mark)

QUESTION TWO (20 MARKS)

a. Generalize the formula $1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}$ to any countable union. Precisely by applying it to two sets and next to three sets, to obtain the general formula.

$$1_{\underset{1\leq j\leq n}{\cup} A_j} = \sum_{1\leq j\leq n} 1_{A_j} + \sum_{r=2}^{n} (-1)^{r+1} \sum_{1<i_1<\ldots<t_r<n} 1_{A_{ij}\ldots A_{ir}}.$$ (9 marks)

b. Show that the class of rectangles S is a semi-algebra. Extend this result to a product of k measurable spaces, $k \geq 2$. (4 marks)

c. Let Ω be a non-empty set and set $\mathcal{A} = \left\{ A \subset \Omega, A \text{ is finite or } A^c \text{ is finite} \right\}$

i. Show that $\mathcal{A}$ is an algebra. (3 marks)

ii. Show that $\mathcal{A}$ is a $\sigma - algebra$ if Ω is finite. (1 mark)

iii. If Ω is infinite, consider an infinite subset $C = \{w_0, w_1, w_2 ....\}$ and set $A_n = \{w_{2n}\}, n \geq 0$. Show that the A_n are in $\mathcal{A}$ and their union is not.

(2 marks)

iv. Conclude that $\mathcal{A}$ is a $\sigma - algebra$ if and only if Ω is finite. (1 mark)

QUESTION THREE (20 MARKS)

a. Prove the assertions:

i. $\left(\bigcup_{n \geq 0} A_n \right) \setminus \left(\bigcup_{n \geq 0} B_n \right) \subseteq \bigcup_{n \geq 0} \left(A_n \setminus B_n \right)$ (4 marks)

ii. $\left(\bigcup_{n \geq 0} A_n \right) \Delta \left(\bigcup_{n \geq 0} B_n \right) \subseteq \bigcup_{n \geq 0} \left(A_n \Delta B_n \right)$ (3 marks)

b. Let $A_n, n \geq 0$ be a sequence of subsets of Ω. Set

$B_0 = A_0, B_1 = A_0^c A_1, B_2 = A_0^c A_1^c A_2, ... B_k = A_0^c A_1^c ... A_{k-1}^c A_k$ for $k \geq 2$.

i. Show that $\bigcup_{n \geq 0} A_n = \sum_{k \geq 0} B_k$. (7 marks)

ii. Interpret the meaning of k for $w \in B_k$, for any $w \in \bigcup_{n \geq 0} A_n$. (2 marks)

c. Let $\Omega = \Omega_1 \times \Omega_2$ be a non-empty product defined by

$\Omega = \Omega_1 \times \Omega_2 = \{(w_1, w_2) \in \Omega, w_1 \in \Omega_1 \text{ and } w_2 \in \Omega_2\}$. Define a rectangle

$A \times B = \left\{ \left(w_1, w_2\right) \in \Omega, w_1 \in A \text{ and } w_2 \in B \right\}$, where $A \subseteq \Omega_1, B \subseteq \Omega_2$.

i. Show that the intersection of a rectangle is a rectangle. (2 marks)

ii. Let $A \times B$ be a rectangle. Show that $\left(A \times B\right)^c = A^c \times B + A \times B^c + A^c \times B^c$.

(2 marks)

QUESTION FOUR (20 MARKS)

a. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ be a measure space. Show that if m is translation-invariant and $m(]0,1]) = 1$ then m is the Lebesque measure λ. (6 marks)

b. Let $(\Omega, \mathcal{A}, m)$ be measure space. Show the following properties:

i. m is non-decreasing: if $A \subset B$, then $m(A) \leq m(B)$. Besides, show, if $m(B)$ is finite, then $m(B \setminus A) = m(B) - m(A \cap B)$. (2 marks)

ii. A measure m finite if and only if $m(\Omega)$ is finite if and only if m is bounded. (2 marks)

iii. A measure m is $\sigma - sub - additive$. In a similar manner show that if $m(\phi) = 0$, and m is non-negative and additive, then it is sub-additive.

(2 marks)

iv. Show that m is continuous in the following sense:

I. Let $(A_n)_{n \geq 0}$ be a non-decreasing sequence of elements of A and set $A = \bigcup_{n \geq 0} A_n$. Then $m(A_n)$ increases to $m(A)$ as $n \to +\infty$. (6 marks)

II. Let $(A_n)_{n \geq 0}$ be a non-increasing sequence of elements of $\mathcal{A}$ and set $\mathcal{A} = \bigcap_{n \geq 0} A_n$ such that $m(A_{n_0})$ for some $n_0 \geq 0$. Then $m(A_n)$ decreases to $m(A)$ as $n \to +\infty$. (2 marks)

QUESTION FIVE (20 MARKS)

a. Prove the algebra generated by a semi-algebra S is the class of all finite sums of its elements. $a(S) = \{A_1 + A_2 + ... + A_k, A_i \in S, k \geq 1\}$. (7 marks)

b. Let C be an algebra and a monotone class. Show that C is a sigma-algebra.

(5 marks)

c. Let D be a $\pi - class$ and a Dynkin system. Show that D is a $\sigma - a\lg ebra$.

(4 marks)

h. Consider a Lebesque measure λ_F on $\mathbb{R}$ where F is a distribution function. Show that for any $x \in \mathbb{R}$, $\lambda_F(\{x\}) = F(x) - F(x-0)$, where $F(x-0)$ is the left-hand limit of F at x. Deduce from this that $\lambda_F(\{x\})$ if and only if F is continuous at x.

(4 marks)