MACHAKOS UNIVERSITY

University Examinations for 2024/2025 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR DEGREE IN
BACHELOR OF SCIENCE IN MATHEMATICS
BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE
BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING
BACHELOR OF EDUCATION (SCIENCE)

SMA 402: FIELD THEORY

DATE: 22ND JANUARY, 2025 **TIME: 2.00 PM – 4.00 PM**

INSTRUCTIONS:
*Answer question **ONE (Compulsory)** and **any other TWO** questions*

QUESTION ONE *(COMPULSORY)* (30 MARKS)

(a) Let E, G be fields. Explain the meaning of the following terms:

(i) a *field extension* E/G, **(3 Marks)**

(ii) an *algebraic field extension* E/G. **(4 Marks)**

(b) Distinguish between a field a of characteristic p and characteristic 0. **(5 Marks)**

(c) Consider the polynomials $f(x) = 2x^5 + 4x^2 + 2$ and $g(x) = 4x^5 + 2x^2 + 4$ in $\mathbb{Z}_8[x]$.

(i) Compute the difference $f(x) - g(x)$ in $\mathbb{Z}_8[x]$. **(3 Marks)**

(ii) Determine the degree of the product $f(x)g(x)$. **(4 Marks)**

(d) Identify the minimal polynomial and determine the basis of the algebraic extension $\mathbb{Q}\left(\sqrt[3]{2} + \sqrt{5}\right)$ over the field $\mathbb{Q}$. **(6 Marks)**

(e) Construct the splitting field of $f(x) = x^2 - 2$ over $\mathbb{Q}$. **(5 Marks)**

QUESTION TWO (20 MARKS)

(a) Let $F[x]$ be a polynomial ring over a field F . Prove that $\alpha \in F[x]$ is invertible if and only if α is a non-zero element of F. **(5 Marks)**

(b) Suppose that K_1, K_2 and K_3 are subfields of a field K. **(5 Marks)**

(i) Prove that $K_1 \cap K_2 \cap K_3$ is a subfield of K.

(ii) Describe the meaning of the *prime* subfield of a field K.

(c) Suppose that F/L and L/K are field extensions such that $[F: L] = n$ and $[L: K] = m$. Prove that F/K is an extension such that $[F: K] = nm$. **(10 Marks)**

QUESTION THREE (20 MARKS)

(a) Distinguish between the term *algebraically closed field* K and that *algebraic closure* $\bar{K}$ of a field K. **(3 Marks)**

(b) Determine the root of $4x + 1$ in $\mathbb{Z}_5[x]$. **(2 Marks)**

(c) Determine whether the following statements are true or false. Justify your answers.

(i) $\mathbb{Q}[\sqrt{2}]$ is an algebraically closed field.

(ii) $\mathbb{R}$ is an algebraically closed field. **(4 Marks)**

(d) Let $p \in \mathbb{Q}$ be a square free rational number (e.g. a prime integer). Describe the field extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$. **(6 Marks)**

(e) Prove that the number $\sqrt[5]{3} + 1$ is algebraic over $\mathbb{Q}$. **(5 Marks)**

QUESTION FOUR (20 MARKS)

(a) Let $f(x) \in K[x]$ be a non-constant polynomial. Explain the meaning of the *splitting field* of $f(x)$ over K. **(3 Marks)**

(b) Determine whether the polynomial $x^2 + 2x + 4$ is irreducible in $\mathbb{Z}_5[x]$. **(4 Marks)**

(c) Let $K \subset L \subset F$ be fields. Show that if $\alpha \in F$ is algebraic over L then α is algebraic over K. **(5 Marks)**

(d) Let $\alpha = \sqrt{3} + 3i \in \mathbb{C}$. Determine the minimal polynomial of α over $\mathbb{Q}$ hence or otherwise construct a basis for the field extension $\mathbb{Q}(\alpha)/Q$. **(4 Marks)**

(e) Suppose that E/F is an extension satisfying $[E: F] = p$ where p is a prime number. Prove that E/F contains no proper intermediate field (i.e. there is no field L such that $F \subsetneq L \subsetneq E$). **(4 Marks)**

QUESTION FIVE (20 MARKS)

(a) Explain the meaning of the term *irreducible polynomial*. **(3 Marks)**

(b) Let F be any field. Prove that any polynomial $f(x) \in F[x]$ of degree 2 or 3 is irreducible if and only if $f(x)$ has no roots in F. **(5 Marks)**

(c) Let F be any field. Prove that for any $n > 0$ then $g(x) = x - 1$ is a divisor of $f(x) = x^n - 1$ in the polynomial ring $F[x]$. **(5 Marks)**

(d) Suppose that F is a field extension of K and $\alpha \in F$ is algebraic over K. Let $f \in K[x]$ be a non-zero polynomial of the least degree such that $f(\alpha) = 0$. Prove that

(i) $f(x)$ is irreducible over K. **(3 Marks)**

(ii) if $p(x) \in K[x]$ is a polynomial such that $p(\alpha) = 0$ then $f(x)|p(x)$. **(4 Marks)**