MACHAKOS UNIVERSITY

University Supplementary Examinations
SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS AND STATISTICS
SMA 304: NUMBER THEORY

DATE:
Answer Question One and any other two questions

Question One - (30 Marks)

(a) Let m be a fixed positive integer. Explain the meaning of the following terms

(i) *congruence modulo* m, **(2 Marks)**

(ii) *reduced residue modulo* m. **(2 Marks)**

(b) Prove that the product of two odd numbers is an odd number. **(4 Marks)**

(c) Let x, y and z be integers. Show that if $x|y$ and $y|z$ then $x|z$.**(4 Marks)**

(d) Let $a \in \mathbb{Z}$ and $b_1, b_2, \dots, b_n$ be a sequence of integers such that $a|b_i$ for each $i = 1,2, \dots, n$. Prove that

$$a|(b_1y_1 + b_2y_2 + \cdots + b_ny_n)$$

for any $y_1, y_2, \dots, y_n \in \mathbb{Z}$.**(5 Marks)**

(e) Let $x, y, z \in \mathbb{Z}$. Prove that if m is any positive integer then:

(i) $x \equiv x(\text{mod } m)$,**(2 Marks)**

(ii) if $x \equiv y(\text{mod } m)$ then $y \equiv x(\text{mod } m)$ and **(3 Marks)**

(iii) if $x \equiv y(\text{mod } m)$ and $y \equiv z(\text{mod } m)$ then $x \equiv z(\text{mod } m)$.**(3 Marks)**

(f) Let $a \in \mathbb{Z}$ be any integer. Verify that $3|a(2a^2 + 7)$. **(5 Marks)**

Question Two - (20 marks)

(a) Given that a, b are non zero integers, prove that $a|b$ and $b|a$ if and only if $a = \pm b$. **(5 Marks)**

(b) Let $m \in \mathbb{Z}$. Prove that either $m^2 = 4k$ or $m^2 = 4k + 1$ for some integer k.**(4 Marks)**

(c) Use the Euclidean algorithm

(i) to find $d = gcd(2327,819)$,**(3 Marks)**

(ii) express $gcd(2327,819)$ in the form $d = 2327x + 819y$.**(3 Marks)**

(d) Let a, b, m be integers with $m > 0$. Prove that if $a \equiv b(mod\ m)$ then $a^n \equiv b^n(mod\ m)$ for any positive integer n.**(5 Marks)**

Question Three - (20 marks)

(a) Distinguish between *prime* numbers and *relatively prime* numbers.**(2 Marks)**

(b) Let a, b, m be integers with $m > 0$ and suppose that $a \equiv b(mod\ m)$.

(i) Prove that $a - x \equiv b - x(mod\ m)$ for any integer x. **(4 Marks)**

(ii) Solve the linear congruence $506x + 6 \equiv 29(mod\ 391)$ **(5 Marks)**

(c) Prove that if m is an integer then $3\ |(m^3 - m)$.**(5 Marks)**

(d) Construct at least two distinct complete residue systems modulo 6. For each of the given complete residue systems determine their corresponding reduced residue systems modulo 6. **(4 Marks)**

Question Four - (20 marks)

(a) Suppose that a, b are non-zero integers.

(i) Explain the meaning of the term *a greatest common divisor* of a and b.**(2 Marks)**

(ii) Suppose that $a = bq + r$ for some integers a, r. Prove that $gcd(a, b) = gcd(b, r)$ **(5 Marks)**

(b) Suppose that $a \in \mathbb{Z}$ is a positive integer greater than 1. Prove that the number $a(a^2 + 2)$ is divisible by 3. **(6 Marks)**

(c) Suppose that a, b are non-zero integers. Prove that

$$gcd(6a - 9b, 9a - 13b)|b.$$

Hence or otherwise show that $gcd(6a - 9,9a - 13) = 1$ for any integer a. **(7 Marks)**

Question Five - (20 marks)

(a) Explain the meaning of the following terms

(i) a *linear Diophantine* equation, **(2 Marks)**

(ii) a pair of *relatively prime* numbers. **(2 Marks)**

(b) Prove that the number $n(n+1)(2n+1)$ is divisible by 6. **(5 Marks)**

(c) Let a, b, c, d, m be non-zero integers with $m > 0$. Suppose that $a \equiv b(mod\ m)$ and $c \equiv d(mod\ m)$. Prove that

(i) $ax - c \equiv bx - d(mod\ m)$ for any integer x,**(3 Marks)**

(ii) $ac \equiv bd(mod\ m)$. **(3 Marks)**

(d) Solve the following linear Diophantine Equations **(5 Marks)**

(i) $111x + 321y = 690$

(ii) $54x + 21y = 91$