MACHAKOS UNIVERSITY

University Supplementary Examinations

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

Examinations for the degree of

Bachelor of education (science & arts), Bachelor of Science (statistics, mathematics)

SMA 300: REAL ANALYSIS I

Date: …………………….. **Time: 2 Hours**

Instructions to Candidates

Answer question ONE and any other TWO questions

Throughout this paper we shall adopt the following notations: $\mathbb{N}$ for the set of Natural numbers, $\mathbb{Z}$ for the set of Integers, $\mathbb{Q}$ for the set of Rational numbers, $\mathbb{Q}^c$ for the of Irrational Numbers and $\mathbb{R}$ for the set of real numbers.

QUESTION 1: 30 MARKS (COMPULSORY)

(a) For all the above sets write down the set inclusions. **(2 Marks)**

(b) With reasons identify which of the above sets are Fields**.** **(2 Marks)**

(c) One property enjoyed by $\mathbb{R}$ but lacking in all the other sets is ***"The completeness property"***

i) State the completeness property of $\mathbb{R}$ **.** **(2 Marks)**

ii) Exhibit an example to show that the set $\mathbb{Q}$ is not complete. **(2 Marks)**

(d) "The concepts of **Neighborhood** of a point, **Open** and **Closed** sets are key in the Topology of the Real line.

i) Define each concept. **(3 Marks)**

ii) Let A and B be open intervals in $\mathbb{R}$. Suppose that A and B are neighborhoods of a point $x \in \mathbb{R}$. Prove that $A \cap B$ is a neighborhood of x . **(4 Marks)**

iii) Find the largest $\boldsymbol{\varepsilon}$ such that A contains an ε- neighborhood of $\boldsymbol{x_0}$ in each case:

1. $\boldsymbol{x_0} = \frac{4}{7}$, $A = [\frac{1}{8}, \frac{2}{3})$. **(1 Marks)**
2. $\boldsymbol{x_0} = -0.625$, $A = (-2.07, \infty)$. **(1 Marks)**

(e) Define what is meant by a convergent sequence hence show from first principles that the sequence: $x_n = 4 + \frac{(-1)^n}{n^2}\ \forall\, n \in \mathbb{N}$ Converges to 4 in $\mathbb{R}$. **(5 Marks)**

(f) Find the limit superior and limit inferior of the sequence $\{a_n\}_{n=1}^{\infty} = \left\{(-1)^n(3 - \frac{1}{n})\right\}_{n=1}^{\infty}$ **(4 Marks)**

(g) Prove that $\lim_{n \to \infty} \frac{3n^2 - 6n}{5n^2 + 4} = 3/5$ **(4 Marks)**

<u>QUESTION 2 (20 MARKS)</u>

a) Define the convergence of a sequence **(2 Marks)**

b) Prove that if $\{x_n\}_{n=1}^{\infty}$ is sequence on real numbers and $\{x_n\}$ converges in $X = \mathbb{R}$ then its limit is unique. **(5 Marks)**

c) Determine the convergence or divergence of the sequence: $\sum_{n=1}^{\infty} \frac{4\sqrt{n}-1}{n^2+2\sqrt{n}}$, $n \in \mathbb{N}$.

(4 Marks)

d) Define what is meant by an irrational number. Given that a and b are rational with $b \neq 0$ and y is an irrational number such that $a - by = z$. Show that z is irrational.

(5 Marks)

e) Show that the set $B = \{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\}$ is closed but not open.

(4marks)

QUESTION 3 (20 MARKS)

a) Define what is meant by infimum and supremum of a subset A of $\mathbb{R}$. **(2 Marks)**

b) The following are subsets of $\mathbb{R}$:

f) $A = \{4 + \frac{(-1)^n}{2n}, n \in \mathbb{N}\}$ ii) $B = (\frac{-3}{11}, 8]$ iii) $C = \{x: x^2 \leq 3\}$.

For each set examine boundedness and determine the following if they exist:

$\boldsymbol{Infimum}$, $\boldsymbol{Supremum}$, $\boldsymbol{Maximum}$ and $\boldsymbol{Minimum}$ values. **(6 Marks)**

a) Give the definition of the concepts of limit superior and limit inferior of any sequence (x_n) of real numbers hence determine the convergence of the following sequence:

$$x_n = \frac{(-1)^{n+1}}{3n-2} \quad \forall\, n \in \mathbb{N}$$ **(3 Marks)**

b) Let $H_n = \left(3 - \frac{1}{n}, 5 + \frac{1}{2n}\right)$ $n \in \mathbb{N}$. Find and comment on:

$i)$ $\bigcup_{n=1}^{\infty} A_n$ **(3 Marks)** ii) $\bigcap_{n=1}^{\infty} A_n$ **(3Marks)**

c) Show that a finite intersection of open sets is open. Hence give a counter example to show that an arbitrary intersection of open sets is not necessarily open. **(6 Marks)**

QUESTION 4 (20 MARKS)

a) Show that the sequence $X_n = 1 + (-1)^n \frac{1}{n^2}$ $n \in \mathrm{N}$ converges to 1. **(5 Marks)**

b) Determine the nth partial sum of the series below. Hence test for its convergence.

$$\sum_{n=1}^{\infty} \frac{3}{n^2 + 3n + 2}$$

(5 Marks)

c) Use integral test to determine the convergence of $\sum_{n=1}^{\infty} \left(\frac{n}{n(n+1)}\right)$ **(5 Marks)**

d) Solve $2\sin(5x) = -\sqrt{3}$ $on[-\pi, 2\pi]$ **(5 Marks)**

QUESTION 5 (20 MARKS)

a) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers. Show that if $a_n \to a$ and $b_n \to b$ then $\lim_{n\to\infty}(a_n + b_n) = a + b$ **(5 marks)**

b) Show that a sequence $\{x_n\}$ of real numbers converges to x if and only if every subsequence of $\{x_n\}$ converges to x. Illustrate by an example. **(5 Marks)**

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c) Show that a sequence $(\frac{1}{2^n})$ in $\mathbb{R}$ is a Cauchy sequence **(5 Marks)**

d) Show that every open interval in $\mathbb{R}$ is a neighborhood of each of its points i.e. is an open set. **(5 Marks)**