MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING

BACHELOR OF SCIENCE IN MATHEMATICS

BACHELOR OF EDUCATION

SMA 300: REAL ANALYSIS I

DATE: 23/3/2021 **TIME: 2.00-4.00 PM**

INSTRUCTIONS

Answer question ONE and any other TWO questions

QUESTION ONE (30 MARKS) (COMPULSORY)

(a) The set of real numbers $\mathbb{R}$ is the union of disjoint subsets $\mathbb{Q}$ and $\mathbb{Q}^c$ of Rational and Irrational numbers respectively.

i) Giving examples, differentiate clearly between the two subsets. (4 marks)

ii) Given that a and b are rational with $b \neq 0$ and s is an irrational number such that $a - bs = t$. Show that t is irrational. (4 marks)

iii) Using (ii) above deduce that $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ is irrational. (2 marks)

(b) The following are basic concepts studied in Real analysis. For each concept give a definition and an example in $\mathbb{R}$.

i) Neighborhood of a point. (2 marks)

ii) Interior point of a set. (2 marks)

iii) Open set. (2 marks)

iv) Limit point of a set. (2 marks)

v) Closed set. (2 marks)

(c) The following are subsets in $\mathbb{R}$

(i) $P = (\frac{1}{6}\ ,\ 3]$ (ii) $Q = \{1 + \frac{(-1)^n}{n}\ , n \in \mathbb{N}\}$ (iii) $S = \{y : y^2 \leq 9\}$.

For each set examine boundedness and determine the following if they exist:

$\boldsymbol{Infimum}$, $\boldsymbol{Supremum}$, $\boldsymbol{Maximum}$ and $\boldsymbol{Minimum}$ values. (6 marks)

(d) Prove that the union of arbitrary collection open sets is open. Exhibit an example. (4 marks)

QUESTION TWO (20 MARKS)

(a) Find the largest $\boldsymbol{\varepsilon}$ such that A contains an ε- neighborhood of $\boldsymbol{x_0}$ in each case:

(i) $\boldsymbol{x_0} = \frac{3}{5},\ A = [\ \frac{6}{7}\ \ ,\frac{13}{11})$ (ii) $\boldsymbol{x_0} = -0.722,\ A = (\ -1.35,\ \infty)$ (4 marks)

(b) Define what is meant by a convergent sequence hence show from first principles that the sequence: $x_{\ n} = 8 + \frac{(-1)^{\ n}}{n^2}\ \forall\, n \in \mathbb{N}$ Converges to 8 in $\mathbb{R}$. (4 marks)

(c) Give the definition of the concepts of limit superior and limit inferior of any sequence (x_n) of real numbers hence determine the convergence of the following sequence:

$$x_n = \frac{(-1)^{n+1}}{5n-2}\ \ \forall\, n \in \mathbb{N}$$

(6 marks)

(d) Show that a sequence $\{x_n\}$ of real numbers converges to x if and only if every subsequence of $\{x_n\}$ converges to x . (3 marks)

(e) Show that $\lim_{n \to \infty} \left(\frac{3n+2}{n+1}\right) = 3$ using any appropriate technique. (3 marks)

QUESTION THREE (20 MARKS)

(a) Let $A_n = \left(1 - \frac{1}{n}\ ,2 + \frac{1}{n}\right)$ $n \in \mathbb{N}$. Find:

(i)$\bigcup_{n=1}^{\infty} A_n$ (ii) $\bigcap_{n=1}^{\infty} A_n$ (4 marks)

(b) Show that a finite intersection of open sets is open. (4 marks)

(c) Let A and B be neighborhoods of a point x. Show that the intersection, $A \cap B$ is a neighborhood of x. (4 marks)

(d) Show that every open interval in $\mathbb{R}$ is a neighborhood of each of its points i.e. is an open set. (4 marks)

(e) Show that if $r = \sqrt{m+1} - \sqrt{m-1}\ \ \forall\, m \in \mathbb{N},$ then r is irrational. (4 marks)

QUESTION FOUR (20 MARKS)

(a) You are given the following sets.

$A = \{\frac{(-1)^n}{n}, n \in \mathbb{N}\}$, $B = \{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\}$. Show that:

i) A is neither open nor closed. (3 marks)

ii) B is closed but not open. (3 marks)

(b) Prove that:

i) The union of any finite number of closed sets is closed. (3 marks)

ii) The intersection of an arbitrary collection of closed sets is closed. Exhibit an example. (4 marks)

(c) Describe what is meant by statement "*The set* $\mathbb{R}$ *of real numbers is complete*". Exhibit an example to illustrate why the set $\mathbb{Q}$ of rational numbers is not complete. (3 marks)

(d) Let S be a bounded subset of the set of real numbers. Prove that sup S and inf S both belong to S if and only if S is closed. (4 marks)

QUESTION FIVE (20 MARKS)

(a) Show that the set Z of integers is countable, hence conclude that the set of rational is also countable. (4 marks)

(b) Define what is meant by a function f being uniformly continuous. (2 marks)

(c) Let $f(x) = x^2 + 1 \ \forall\, x \in \mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = 5$. (3 marks)

(d) Prove that the series $\sum_{n=1}^{\infty}\left[\frac{1}{n^p}\right]$ converges if $p < 1$ and diverges if $p \leq 1$. (4 marks)

(e) Apply the indicated test of convergence:

(i) $\sum \frac{\sqrt{n}}{n^2+1}$ (comparison test). (2 marks)

(ii) $\sum \frac{1}{n^n}$ (Root test). (2 marks)

(iii) $\sum \frac{1}{n^2+1}$ (Integral test). (3 marks)