MACHAKOS UNIVERSITY

University Examinations for 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR SPECIAL/SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (ANALYTICAL CHEMISTRY)

BACHELOR OF SCIENCE (ACTUARIAL SCIENCE)

BACHELOR OF ECONOMICS AND STATISTICS

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION

SMA 300: REAL ANALYSIS I

DATE: 7/3/2023 **TIME: 2:00 – 4:00 PM**

INSTRUCTIONS

1. Answer **question one** and **any other two** questions
2. Write the name of the unit and registration number on each page of your answer sheet.

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Describe what is meant by statement "The set $\mathbb{R}$ of real numbers is complete". Exhibit an example to illustrate why the set $\mathbb{Q}$ of rational numbers is not complete. (3 marks)

b) Explain what is meant by upper bound, infimum of a set and a countable set. (3 marks)

c) Show that the sequence $\left\{\frac{1}{2^n}\right\}$; $n \in \mathbb{N}$ is bounded and determine its supremum and infimum. (4 marks)

d) Determine the convergence or divergence of

$\sum_{n=1}^{\infty} \frac{4\sqrt{n}-1}{n^2+2\sqrt{n}}$ (4 marks)

e) Let A and B be neighborhoods of a point$x$. Show that the intersection $A \cap B$ is a neighborhood of$x$. (4 marks)

f) Prove that if a sequence converges then its limit is unique. (4 marks)

g) Let $A_n = (2 - \frac{1}{2n}, 5 + \frac{1}{n})$ $n \in \mathbb{N}$. Find:

(i) $\bigcup_{n=1}^{\infty} A_n$ (ii) $\bigcap_{n=1}^{\infty} A_n$ (4 marks)

h) Prove that convergence of a sequence implies boundedness. Use an example to show that the converse need not be true. (4 marks)

QUESTION TWO (20 MARKS)

a) Let $X = \mathbb{R}$. Describe what is meant by

i) Neighborhood of a point $x \in X$. (2 marks)

ii) An open set $O \in X$. (2 marks)

b) Given that $A = (3,7)$ find the exterior of A. (2 marks)

c) Consider the set $A = (1,10) \cup \{11\}$ .Show that 11 is the isolated point of A. (3 marks)

d) Prove that the set of rational$\mathbb{Q}$ has no interior point (4 marks)

e) Use limit comparison test determine the convergence of

$$\sum_{k=0}^{\infty} \left(\frac{1}{2^k - 5}\right)$$ (4 marks)

f) Show that the set $Y = \{\sin x \,|0 \le x \le 2\pi, x \in \mathbb{R}\}$ is bounded. Determine the supremum and infimum. (3 marks)

QUESTION THREE (20 MARKS)

a) Define what is meant by a convergent sequence hence show from first principles that the sequence: $y_n = 3 + \frac{(-1)^n}{n^2}$ Converges to 3 in $\mathbb{R}$. (4 marks)

b) Determine the nth partial sum of the series below. Hence test for its convergence.

$$\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$$ (5 marks)

c) Show that $\lim_{n \to \infty} (1 - \frac{1}{2^n}) = 1$ (2 marks)

d) Show that $f(x) = 2x + 1$ is Riemann integrable on $[1,2]$ and hence find $\int_1^2 (2x + 1)dx$, (6 marks)

e) Show that a finite intersection of open sets is open. (3 marks)

QUESTION FOUR (20 MARKS)

a) Explain the following terms

i) Monotonic increasing sequence. (1 mark)

ii) Cauchy sequence. (2 marks)

b) Test for convergence of the following series

$$(i)\ \sum_{n=1}^{\infty}\frac{10^n}{n4^{2n+1}} \qquad (ii)\ \sum_{n=0}^{\infty}\left(1+\frac{1}{n}\right)^{n^2} \qquad (iii)\ \sum_{n=1}^{\infty}\frac{(2n)!}{2^n n!}$$

(9 marks)

c) Show that from the first principle that limit $\lim_{x\to 2}(x^2-4x+2)=-2$ (4 marks)

d) Determine whether the function $f(x)=\begin{cases}-x^2+4 & x\le 3\\ 4x-8 & x>3\end{cases}$ is continuous at$x=3$.

Justify your answer. (4 marks)

QUESTION FIVE (20 MARKS)

a) Show that $f(x)=x^2+1$ is Riemann integrable on $[0,1]$ and hence find$\int_0^1(x^2+1)dx$

(10 marks)

b) Prove that every convergent sequence is Cauchy. (4 marks)

c) Let $\mathbb{R}$be the set of real numbers. Define a function $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ by $d(x,y)=|x-y|\forall x,y\in\mathbb{R}$. Show that d is a metric space. (6 marks)