MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

SMA 465: MEASURE AND PROBABILITY

DATE: 16/12/2021 **TIME: 2.00-4.00 PM**

INSTRUCTION:
Answer Question One and Any Other Two Questions

QUESTION ONE COMPULSORY (30 MARKS)

a) Define the following terms as used in measure and probability:
   i. subset
   ii. universal set
   iii. power set (3 marks)

b) Differentiate between:
   i. upper limit and lower limit of sequences of sets.
   ii. bijective (one-one) function and one-many function (4 marks)

c) State:
   i. Lebesgue measure. (2 marks)
   ii. the convex function. (2 marks)

d) Show that $E(X \pm Y) = E(X) \pm E(Y)$ (5 marks)

e) Give three properties of the characteristic functions. (3 marks)

f) Define:
   i. a point function
   ii. set function (2 marks)

g) Determine whether the sequence $\{A_n, n = 1, 2, \dots\}$ is monotone increasing or decreasing and find its limit, where

$A_n = \{w: 2 + \frac{2}{3n} < w < 4 - \frac{1}{n}$ (5 marks)

h) State the:

i. Borel-Cantelli lemma. (2 marks)

ii. the Central Limit Theorem (2 marks)

QUESTION TWO (20 MARKS)

a) Give the definition of an indicator function and state the four properties of an indicator function. (6 marks)

b) Let A be a set. Show that the power set of A has 2^n. (4 marks)

c) Given the set of arbitrary sets $A_1, A_2, \dots, A_n$. Show that there exists a set of disjoint sets $B_1, B_2, \dots, B_n$such that $\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i$. (5 marks)

d) Show that $f^{-1}(x)\ for\ f(x) = 3x^2$ does not exist, but $f^{-1}(x)\ for\ f(x) = \{3x^2 | x \geq 0\}$ exists. (5 marks)

QUESTION THREE (20 MARKS)

a) Define:

i. $\sigma - field$

ii. Borel field (2 marks)

b) Differentiate between:

i. a partition and a monotone field. (2 marks)

ii. finite field and $\sigma -$ field (2 marks)

c) A σ-field is a monotone field, and conversely a monotone field is a σ-field. Prove. (6 marks)

d) Consider the measurable space $(\Omega, \digamma, \mu)$ and suppose $f, g \in \digamma$. Suppose $\Omega \to \mathbb{R}$, and $\alpha \in \mathbb{R}$. Show that;

i. $\alpha f \in \digamma$

ii. $f + g \in \digamma$ (8 marks)

QUESTION FOUR (20 MARKS)

a) Define:

i. an event

ii. random variable (2 marks)

b) Differentiate between:

i. a sample space and an experiment. (2 marks)

ii. the classical concept and the axiomatic concept of probability. (2 marks)

c) State and prove Jensen's inequality. (5 marks)

d) Show that $E|X+Y|^r \leq C_r E|X|^r + C_r E|Y|^r$ where

$$C_r = \begin{cases} 1 & if\ r \leq 1 \\ 2^{r-1} & if\ \ r \geq 1 \end{cases}$$ (9 marks)

QUESTION FIVE (20 MARKS)

a) Define convergence in probability. (2 marks)

b) State:

i Minkowski's inequality. (1 mark)

ii Chebychev's inequality (2 marks)

c) Let X be an arbitrary random variable and let g be an even non-decreasing non-negative Borel function in the interval [0, ∞). Show that, for every $a > 0$

$$\frac{Eg(x)-g(a)}{\sup g(x)} \leq P[|X| > a] \leq \frac{Eg(x)}{g(a)}$$ (8 marks)

d) Show that $X_n \xrightarrow{P} 0$ iff $E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0\ as\ n \to \infty$. (7 marks)