MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECOND YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 204: ALGEBRAIC STRUCTURES

DATE: 7/12/2021 **TIME: 8:30 – 10:30 AM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Explain the meaning of the following terms

i. a commutative binary operation on a set A, (2 marks)

ii. an associative binary operation on a set A. (2 marks)

b) Let $\mathbb{Z}$ be the set of all integers and let $*$ be the binary operation defined on $\mathbb{Z}$ by $x * y = x + y + 1$ for all $x, y \in \mathbb{Z}$.

i. Prove that $*$ is an associative binary operation. (2 marks)

ii. Prove that -1 is the identity element under the binary operation $*$ (2 marks)

iii. Determine the inverse of $7 \in \mathbb{Z}$ under the binary operation $*$ (2 marks)

c) Let $\oplus$ denote operation of addition modulo 7 on the set $\mathbb{Z}_7$. Construct the Cayley table for the operation $\oplus$ on $\mathbb{Z}_7$. (5 marks)

d) Consider $\mathbb{Z}_{22}$ under multiplication modulo 22 (denoted by $\otimes$).

i. Is $\langle \mathbb{Z}_{22}, \otimes \rangle$ a group? Justify your answer. (3 marks)

ii. Determine the elements of U_{22} and construct the its Cayley table. (4 marks)

iii. Let H be the cyclic subgroup of U_{22} generated by $13 \in U_{22}$. Determine the elements of H. (4 marks)

e) Let G be a cyclic group. Prove that G is commutative. (4 marks)

QUESTION TWO (20 MARKS)

a) Let $\odot$ be a binary operation defined on $\mathbb{Z}$ by $a \odot b = ab - a - b$.

i. Show that $\odot$ is a commutative binary operation. (3 marks)

ii. Is $\odot$ an associative binary operation? Justify your answer. (3 marks)

b) Let G be a group.

i. State the cancellation laws in group theory. (2 marks)

ii. Prove that if G is a group and $a, b \in G$ are elements of G such that $ab^2 = a^3b$ then $ba = a^3$. (5 marks)

c) Suppose that $(G,*)$ is a group and $a \in G$ be an element such that $a^2 = a$. Prove that $a = e$ where e is the identity element in G. (2 marks)

d) Let $M(2, \mathbb{R})$ be the group of all 2×2 matrices under usual matrix addition. Let U be the subset of $M(2, \mathbb{R})$ given by

$$U = \left\{A \in M(2, \mathbb{R}) \text{ such that } A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \text{ for some } a, b, d \in \mathbb{R}\right\}$$

Determine whether U is a subgroup of $M(2, \mathbb{R})$. (5 marks)

QUESTION THREE (20 MARKS)

a) State the Lagrange's theorem. (2 marks)

b) Let $G = \langle \mathbb{Z}, + \rangle$ be the group of integers under the standard addition.

i. Let K be the subset consisting of all odd numbers in $\mathbb{Z}$. Is K a subgroup of G? Justify. (2 marks)

ii. Let H be the subset consisting of all even numbers in $\mathbb{Z}$. Prove that H is a subgroup of G. (3 marks)

iii. Determine all the distinct left cosets of H. (3 marks)

c) Let G be a group

i. Show that if $a, b \in G$ are elements such that $a^2b^2 = a^4b$ then $ba = a^3$. (3 marks)

ii. Prove that if $(ab)^2 = a^2b^2$ for all $a, b \in G$ then G is a commutative group.(3 marks)

d) Suppose that G is a group of order 105 and suppose that H is a non-trivial subgroup of G. What are the possible values of $o(H)$? Justify your answer. (4 marks)

QUESTION FOUR - (20 MARKS)

a) Explain the meaning of the following terms:

i. a *right coset* of a subgroup, (2 marks)

ii. the *kernel* of a group homomorphism. (2 marks)

b) Let $U_{14} = \{1,3,5,9,11,13\}$ be the group of invertible elements in $\mathbb{Z}_{14}$ under multiplication modulo 14. Let H be the cyclic subgroup of U_{14} generated by 13. Determine all the distinct left cosets of H. (6 marks)

c) Let $G_1 = \langle \mathbb{R}, + \rangle$ be the group of real numbers under standard addition of real numbers and $G_2 = \langle \mathbb{R}^+, \times \rangle$ be the group of all positive real numbers under standard multiplication. Show that

i. the function $f: G_1 \to G_2$ defined by $f(x) = \exp(x)$ for any $x \in G_1$ is a group homomorphism. (Here, $\exp(x) = e^x$ is the usual exponential function). (5 marks)

ii. the function $g: G_2 \to G_1$ defined by $g(a) = \ln(a)$ for any $a \in G_2$ is a group homomorphism. (Here $\ln(a)$ is the usual natural logarithm function). (5 marks)

QUESTION FIVE (20 MARKS)

a) Explain the meaning of the following terms:

i. an *integral domain*, (2 marks)

ii. the *characteristic* of a ring, (2 marks)

b) Suppose that R is a set endowed with an additive binary $+$ and a multiplicative binary operation denoted by $\times$. What properties must the triple $(R, +, \times)$ satisfy in order for it to be a ring? (8 marks)

c) Let $\mathbb{Q}$ denote the set of rational numbers. Define the set

$$\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$$

d) If we define addition and multiplication on $\mathbb{Q}(\sqrt{3})$ as follows

$$(a + b\sqrt{3}) \oplus (c + d\sqrt{3}) = (a + c) + (b + d)\sqrt{3}$$
$$(a + b\sqrt{3}) \otimes (c + d\sqrt{3}) = (a + b\sqrt{3})(c + d\sqrt{3})$$

Show that $(\mathbb{Q}(\sqrt{3}), \oplus, \otimes)$ is a ring. (8 marks)