MACHAKOS UNIVERSITY

University Examinations 2017/2018

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE IN MATHEMATICS

BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

SMA303: RING THEORY

DATE:20/12/2017 **TIME: 11.00-1.00 PM**

INSTRUCTIONS: Answer questions ONE and any other TWO questions.

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Define the following terms as used in ring theory

i) Division ring (2 marks)

ii) Ideal (2 marks)

iii) Characteristic of a ring (2 marks)

iv) Maximal ideal (2 marks)

b) Show that if I is an ideal of a ring R, then a map that maps a ring to a quotient ring

$\gamma: R \to \frac{R}{I}$ given by $\gamma(a) = a + I$ for $a \in R$ is a ring homomorphism. (3 marks)

c) Show that a field contains no proper ideals. (4 marks)

d) Divide $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$ by $g(x) = x^2 - 2x + 3$ in $Z_5[x]$ expressing your answer in the form $f(x) = g(x)\,q(x) + r(x)$ (4 marks)

e) List all the units in Z_{12} and give the inverse of each. (3 marks)

f) Define a principal ideal and hence show that all ideas in the ring Z of integers are principal ideals. (5 marks)

g) Show that if R is a ring with unity e then this unity e is unique. (3 marks)

QUESTION TWO (20 MARKS)

a) If $f: R \to S$ is a ring homomorphism of a ring R into a ring S, show that if 0_R is the zero element in R then $f(0_R) = 0_S$ in S (3 marks)

b) Let $f: R \to S$ be a homomorphism of a ring R into a ring S. show that;

i) If $a \in R$ then $F(-a) = -f(a)$ (2 marks)

ii) The set $\{a \in R | f(a) = 0_S\}$ is an ideal in R called the kernel of f and denoted by $ker\ f$ (3 marks)

iii) If 1_R is the multiplicative identity in R then $f(1_R)$ is the unity in S. (1 mark)

iv) f is one to one iff $ker\ f = \{0\}$ (4 marks)

c) Let F be a field. Show that an ideal $\langle p(x) \rangle \neq 0$ of $F(x)$ is maximal iff $p(x)$ is irreducible over F. (7 marks)

QUESTION THREE (20 MARKS)

a) Show that every finite integral domain is a field. (5 marks)

b) Show that if R is a ring with unity then R has characteristic $n > 0$ iff n is the smallest positive integer such that $n.1 = 0$ (5 marks)

c) Let S denote the ring of ordered pairs of real numbers $\mathbb{R}$ with addition and multiplication defined by;

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

Show that S is a field. (10 marks)

QUESTION FOUR (20 MARKS)

a) If F is a field show that every non-constant polynomial $f(x) \in F[X]$ can be factorized in $F[X]$ into a product of irreducible polynomials being unique except for order and for units (8 marks)

b) State the Einstein's irreducibility criterion and hence show that $17x^3 + 5x^2 + 15x - 5$ is irreducible over $\mathbb{Q}$ (the ring of rational numbers). (5 marks)

c) Define a prime ideal and hence show that if R is a commutative ring with unity and I an ideal of R then $\frac{R}{I}$ is an integral domain iff I is a prime ideal (5 marks)

d) Let D be an integral domain, explain what is meant by elements $a, b \in D$ are associates in D (2 marks)

QUESTION FIVE (20MKS)

a) Differentiate between a field and an integral domain and hence give an example of an integral domain which is not a field. (5 marks)

b) Let R be a commutative ring with unity 1 show that I is maximal ideal iff $\frac{R}{I}$ is a field. (8 marks)

c) Show that in the ring Z_n the zero divisors are those elements in Z_n that are not relatively prime to n. (6 marks)

d) Differentiate between a unit and a unity in a ring. (1 mark)