MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF EDUCATION (SCIENCE)

SMA 406: FUNCTIONAL ANALYSIS

DATE: 2/9/2022 **TIME: 8.30-10.30 AM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (30 MARKS)

a) Explain the following terms as used in functional analysis

i. Cauchy sequence (2 marks)

ii. Linearly independent (2 marks)

iii. Finite dimensional (1 mark)

iv. A basis (1 mark)

b) Explain what is a norm and give an example of a normed space (5 marks)

c) Let X be an n-dimensional vector space, then prove that any proper subspace Y of X has dimension less than n. (5 marks)

d) Differentiate between orthogonal and orthonormal sets (2 marks)

e) Prove that if X is a normed space, $||x|| - ||y|| \leq ||x - y||, \forall x, y \in X$ (3 marks)

f) Let X be a normed space and let $x_0 \neq 0$ be any element of X, then prove that there exists a bounded linear functional $\overline{f}$ on X such that $||\overline{f}|| = 1, ||f||_z = ||x_0||$ (5 marks)

g) Show that if X is a Hilbert space then $S, T \in L(X), (S+T)^* = S^* + T^*$ (4 marks)

QUESTION TWO (20 MARKS)

a) Define a Hilbert space (2 marks)

b) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a Banach space and $s_n = x_1+\ldots+x_n$. Prove that if $\sum_{n=1}^{\infty}||x_n|| < \infty$ then the series $\sum_{n=1}^{\infty}x_n$ converges and if we denote $\lim_{n\to\infty} s_n$ by $\sum_{n=1}^{\infty}x_n$ then $||\sum_{n=1}^{\infty}x_n|| \geq \sum_{n=1}^{\infty}||x_n||$ (6 marks)

c) Explain what is a vector space and give two examples of a vector space (8 marks)

d) Show that the mapping $T: C[a,b] \to K$ given by $Tf = f(\frac{a+b}{2})$ for all $f \in C[a,b]$ is a linear transformation (4 marks)

QUESTION THREE (20 MARKS)

a) Prove that if Y is complete, then $L(X,Y)$ is also complete (8 marks)

b) Prove that the spaces l^p are Banach spaces for $1 \leq p \leq \infty$ (10 marks)

c) Show that if X is a Hilbert space then $S,T \in L(X), (ST)^* = T^*S^*$ (2 marks)

QUESTION FOUR (20 MARKS)

a) Explain the continuity of functions between normed spaces (3 marks)

b) Show that $Re(\langle x,y\rangle) = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$ and $Im(\langle x,y\rangle) = Re(\langle x,iy\rangle)$(5 marks)

c) Explain what in an inner product (4 marks)

d) Prove that for a bounded linear operator, $||Tx|| \leq M||x||, \forall x \in X$, there that of all constant M, there is a smallest one, given by $||T|| = \sup_{||x|| \leq 1}$ (3 marks)

e) Prove that if X is an inner product space, then for all $x,y \in X, |\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$ and $|\langle x,y\rangle|^2 = \langle x,x\rangle\langle y,y\rangle$ if x and y are linearly independent. (5 marks)

QUESTION FIVE (20 MARKS)

a) Prove that the set $L(X,Y)$ of bounded linear operators from normed spaces X and Y with addition and scalar multiplication defined by

$$(T+S)x = Tx + Sx, x \in X,$$

$(\alpha T)x = \alpha Tx, x \in X, \alpha \in \kappa$ is a vector space. Furthermore, show that the mapping $T \to ||T||$ is a norm on this space. (8 marks)

b) Show that sequence $(x_n)_{n\in\mathbb{N}}$ converges to x (3 marks)

c) Explain what is the adjoint of T (3 marks)

d) Show that if X is a Hilbert space, then

i. $S,T \in L(X), (T^*)^* = T$ (3 marks)

ii. $S,T \in L(X), ||T^*|| = ||T||$ (3 marks)