MACHAKOS UNIVERSITY

University Examinations for 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (COMPUTER SCIENCE)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION

SMA 406: FUNCTIONAL ANALYSIS

DATE: 13/8/2021 **TIME: 8:30 – 10:30 AM**

INSTRUCTIONS

1. Answer **question one** and **any other two** questions
2. Write the name of the unit and registration number on each page of your answer sheet.

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Explain what is a Cauchy sequence (3 marks)

b) Prove that if $B: X \to Y$ and $A: Y \to Z$ are bounded linear operator, then the composition $AB: X \to Z$ is a bounded operator, and there holds $||AB|| \leq ||A||||B||$ (3 marks)

c) Let X and Y be normed spaces over $K(\mathbb{R} or \mathbb{C})$. Let $T: X \to Y$ be linear transformation. Prove that T is continuous at 0 and there exists a number m such that for all $x \in X, ||Tx|| \leq m||x||$ (5 marks)

d) Let V be a vector space, and let $\{V_n | n \in \mathbb{N}\}$ be set of subspaces of V. Prove that $\overset{n=1}{\underset{\infty}{\cap}} V_n$ is a subspace of V. (5 marks)

e) Explain what is a norm and give two examples of normed spaces (5 marks)

f) The set $\lambda(X, Y)$ of bounded linear operators from normed spaces X and Y with addition and scalar multiplication defined by

$$(T + S)x = Tx + Sx, x \in X,$$

$(\alpha T)x = \alpha Tx, x \in X, \alpha \in \kappa$ is a vector space. Prove that the mapping $T \to ||T||$ is a norm on this space (5 marks)

g) Show that sequence $(x_n)_{n\in\mathbb{N}}$ converges to x (2 marks)

h) Differentiate between orthogonal and orthonormal sets (2 marks)

QUESTION TWO (20 MARKS)

a) Let $\{x_1, x_2, \ldots\}$ be a linearly independent subset of an inner product space X . Define $u_1 = \frac{x_1}{||x_1||}$ and $u_n = \frac{x_n - \langle x_n, u_1\rangle u_1 - \ldots - \langle x_n, u_{n-1}\rangle u_{n-1}}{||x_n - \langle x_n, u_1\rangle u_1 - \ldots - \langle x_n, u_{n-1}\rangle u_{n-1}||}$ for $n \geq 2$, then prove that $\{u_1, u_2, \ldots\}$ is an orthonormal set in X and for $n \in \mathbb{N}$, $span\{x_1, \ldots, x_n\} = span\{u_1, \ldots, u_n\}$ (8 marks)

b) Let $n \in \mathbb{Z}$, and let $T_n \in C[0,1]$ denote the trigonometric polynomial defined by $T_n(x) = e^{2\pi inx}, x \in [0,1]$. Show that the set $\{T_n | n \in \mathbb{Z}\}$ is orthonormal sequence in C[0,1] with respective to the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)dx$ (4 marks)

c) Prove that if Y is complete, then $\lambda(X, Y)$ is also complete (8 marks)

QUESTION THREE (20 MARKS)

a) Define a Hilbert space (2 marks)

b) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a Banach apace and $s_n = x_1 + \ldots + x_n$. Prove that if $\sum_{n=1}^{\infty} ||x_n|| < \infty$ then the series $\sum_{n=1}^{\infty} x_n$ converges and if we denote $\lim_{n\to\infty} s_n$ by $\sum_{n=1}^{\infty} x_n$ then $|| \sum_{n=1}^{\infty} x_n|| \geq \sum_{n=1}^{\infty} ||x_n||$ (6 marks)

c) Explain what is a vector space and give two examples of vector spaces (6 marks)

d) Show that the mapping $T: C[a, b] \to K$ given by $Tf = f(\frac{a+b}{2})$ for all $f \in C[a, b]$ is a linear transformation (6 marks)

QUESTION FOUR (20 MARKS)

a) Consider the squaring map $S: C[a, b] \to C[a, b]$ defined as follows$(S(u))(t) = ((t))^2, t \in [a, b], u \in C[a, b]$. Show that the map is not linear but it is continuous. (10 marks)

b) Prove that the spaces l^p are Banach spaces for $1 \leq p \leq \infty$ (10 marks)

QUESTION FIVE (20 MARKS)

a) Explain what is a continuous function and show that a map $T: X \to Y$ is continuous (5 marks)

b) Show that $Re(\langle x, y\rangle) = \frac{1}{4}(||x + y||^2 - ||x - y||^2)$ and $Im(\langle x, y\rangle) = Re(\langle x, iy\rangle)$ (5 marks)

c) Explain what in an inner product and state Hann Banach theorem (5 marks)

d) Prove that if X is an inner product space, then for all $x, y \in X, |\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$ and $|\langle x, y\rangle|^2 = \langle x, x\rangle\langle y, y\rangle$ if x and y are linearly independent (5 marks)