MACHAKOS UNIVERSITY

University Examinations 2017/2018

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR BACHELOR OF EDUCATION (SCIENCE)

SMA 403: GALOIS THEORY

DATE: 5/12/2017 **TIME: 8:30 – 10:30 AM**

INSTRUCTIONS

Answer question ONE and any other TWO questions

QUESTION ONE : (COMPULSORY) (30 MARKS)

a) Briefly explain what you understand by the term ***Galois group.*** (3 marks)

b) i) State the ***fundamental theorem of Galois theory***. (3 marks)

ii) By using constructing lattice diagrams of the subgroups of Galois group and subfields of the splitting field verify (i) for the polynomial $f(x) = x^2 + 1$ over Q (3 marks)

c) Describe what is meant by ***Cyclotomic extension*** E of a field F . (3 marks)

d) Using (c) above identify the cyclotomic extension of $f(x) = x^6 - 1$, hence write down the corresponding ***cyclotomic polynomial***. (5 marks)

e) Find the ***splitting field*** E of the polynomial $f(x) = x^4 - 2x^2 - 3$ over the field Q.

(4 marks)

f) Using (e) above Construct the Galois group $G(E/Q)$. (4 marks)

g) Using cayley theorem show that $G(E/Q)$ is isomorphic to a subgroup (***Klein IV group***) of S_4 . (5 marks)

QUESTION TWO (20 MARKS)

a) Describe what is meant by a ***separable polynomial***. (2 marks)

b) Let E be the splitting field of the polynomial $f(x) = x^4 - 10\,x^2 + 21$ over Q .

i. Determine the roots of this polynomial. (4 marks)

ii. Identify E, splitting field of the polynomial. (2 marks)

iii. The extension field E field may considered as a vector space. Identify the basis elements of this vector space and state its dimension. (4 marks)

iv. Construct the group of the roots of the equation $x^4 - 10x^2 + 21 = 0$, the Galois group, by clearly describing its elements. (4 marks)

v. Construct the group table for part (iv) above. (4 marks)

QUESTION THREE (20 MARKS)

a) Differentiate between ***a homomorphism*** and ***an automorphism***. (3 marks)

b) Let $\{\mu_i : i \in N\}$ be a collection of automorphism of a field F, show that $F(\mu_i) = \{\, a \in F : \mu_i(a) = a \text{ for all } \mu_i\}$ is a subfield of F. (5 marks)

c) Let E be a field extension of F and $f(x)$ be a polynomial in $F(x)$. Show that any automorphism in $G(E/F)$ defines a permutation of the roots of $f(x)$ that lie in E. (6 marks)

d) Determine the solvability by radicals and compute the Galois group of the polynomial $f(x) = x^3 - 5$. (6 marks)

QUESTION FOUR (20 MARKS)

a) Briefly explain what is meant by ***an extension*** K of a field F by radicals. (2 marks)

b) Let $E = Q(\sqrt{2}, \sqrt{11})$ be an extension of the field Q by radicals.

(i) Describe and identify the elements of the Galois group $G(Q(\sqrt{2}, \sqrt{11})/Q)$ (4 marks)

(ii) Demonstrate that $G(Q(\sqrt{2}, \sqrt{11})/Q)$ is isomorphic to a subgroup of the symmetric group S_4 . (You may use the numbers 1,2,3,4) (4 marks)

(iii) Let $u = \sqrt{2} + \sqrt{11}$ be a primitive root of the minimal polynomial in $Q(\sqrt{2}, \sqrt{11})$ over Q. Using (i) above determine this minimal polynomial. (4 marks)

c) Find a primitive element in the splitting field in each of the following polynomials over $Q[x]$.

(i) $x^4 - 18x^2 + 65$ (ii) $x^3 - 7$ (6 marks)

QUESTION FIVE (20 MARKS)

a) Consider the Field $E = Q(\sqrt{3}, \sqrt{5}, \sqrt{7})$, find the index of E over Q. (4 marks)

b) Let $\tau_3 = \gamma_{\sqrt{3},\ -\sqrt{3}}$, $\tau_5 = \gamma_{\sqrt{5},\ -\sqrt{5}}$, $\tau_7 = \gamma_{\sqrt{7},\ -\sqrt{7}}$

Compute the indicated elements of E.

i) $\tau_3(\sqrt{3} + \sqrt{15})$

ii) $(\tau_3\tau_5)(\sqrt{3} + 5\sqrt{7})$

iii) $(\tau_5\tau_7)\left(\frac{\sqrt{5}-3\sqrt{7}}{2\sqrt{7}-\sqrt{10}}\right)$ (6 marks)

c) Using (a) find the fixed field of the automorphism(s) of E for :

(i)$(\tau_7\tau_3)$ (ii) $(\tau_7\tau_5\tau_3)$ (iii) $(\tau_5{\tau_7}^2)$ (6 marks)

d) Describe the subgroup K of $G(E\backslash Q)$ generated by the elements τ_3 , τ_5, τ_7 and construct the group table. (4 marks)