



MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FOURTH YEAR SPECIAL/SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN MATHEMATICS

BACHELOR OF EDUCATION

SMA 406: FUNCTIONAL ANALYSIS

DATE: 23/7/2019

TIME: 8:30 – 10:30 AM

INSTRUCTIONS TO CANDIDATES

(a) Answer ALL the questions in Section A and ANY TWO Questions in Section B

QUESTION ONE

- a) Define the following:
- (i) Metric space (2 marks)
 - (ii) Normed space and Banach space . (2 marks)
 - (iii) Equivalent norms . (2 marks)
- b) Show that a subspace Y of a Banach space X is complete if and only if Y is closed in X . (5 marks)
- c) Given an operator $T: X \rightarrow Y$ defined by $Tx=3x$.show that T is linear. (3 marks)
- d) Show that :
- (i) $|||x|| - ||y||| \leq ||x - y||$ for all $x, y \in X$. (2 marks)
 - (ii) $|||x|| - ||y||| \leq ||x + y||$ for all $x, y \in X$. (2 marks)
- e) Show that $f(x) = \int_a^b x(t) dt$, $x \in C[a, b]$ is linear and bounded functional on $C[a, b]$. (6 marks)
- f) Show that every convergent sequence in a normed space is a Cauchy sequence. (6 marks)

QUESTION TWO (20 MARKS)

- a) Prove the continuous mapping theorem which states that a mapping T on a Metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X . (8 marks)
- b) Show that the vector space $X = \mathbb{R}^2$ is a normed space with respect to the following norm
- $$\|p\| = (x^2 + y^2)^{1/2} \quad \text{for all } x, y \in \mathbb{R}^2. \quad (6 \text{ marks})$$
- c) Let (x_n) be a weakly convergent sequence in a normed space X , say $x_n \rightarrow x$. show that weak limit in x_n is unique. (6 marks)

QUESTION THREE (20 MARKS)

- a) Show that the linear space $\mathbb{R}^n$ with $\langle, \rangle$ defined for arbitrary vectors $x = (x_1, x_2, x_3, \dots, x_n)$, $y = (y_1, y_2, y_3, \dots, y_n)$ in $\mathbb{R}^n$ by $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ is an inner product. (4 marks)
- b) Show that an inner product on E defines a norm on E given by $\|x\| = \sqrt{\langle x, x \rangle}$. (6 marks)
- c) If in an inner product space $x_n \rightarrow x$ and $y_n \rightarrow y$ show that $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$. (5 marks)
- d) Show that the space l_p with p not equal to zero is not an inner space product space and hence not a Hilbert space. (5 marks)

QUESTION FOUR (20 MARKS)

- a) Let X, Y be vector spaces and $T: D(T) \rightarrow R(T)$ be a linear operator with $D(T) \subset X$ and $R(T) \subset Y$. Show that :
- (i) The inverse of T exist if and only if $Tx=0$ implies $x=0$. (5 marks)
- (ii) If inverse of T exist, it is a linear operator. (5 marks)
- b (i) Show that the linear space $C[0,1]$ with the function $\langle, \rangle$ defined for arbitrary $f, g \in C[0,1]$ by $\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} dx$ is an inner product space. (6 marks)
- (iii) Verify that $\langle x, 0 \rangle = 0$ for all $x \in X$. (4 marks)

QUESTION FIVE (20 MARKS)

- a) Show that any closed subset of a compact metric space is compact. (7 marks)
- b) Show that any sequentially compact subset of a metric space is bounded and closed. (7 marks)
- c) If (X, d) is a metric space with the discrete metric, show that the induced topology consists of all the subsets of X . (6 marks)