



MACHAKOS UNIVERSITY

University Examinations 2022/2023

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR SPECIAL/SUPPLEMENTARY EXAMINATION

SMA 303: RING THEORY

DATE: 6/3/2023

TIME: 2.00 – 4.00 PM

INSTRUCTIONS

Answer Question 1 (compulsory) and any two questions

QUESTION ONE - (30 MARKS)

(a) Explain the meaning of the following terms:

- (i) a *non-commutative ring*, (2 marks)
- (ii) an *integral domain*, (2 marks)
- (iii) a *right ideal* in a ring R . (2 marks)

(b) Show that if R is a ring and $a, b, c, d \in R$ then $(a + b)(c + d) = ac + ad + bc + bd$. Justify your answer. (4 marks)

(c) Let $\mathbb{Q}[\sqrt{2}]$ denote the set $\mathbb{Q}[\sqrt{2}] = \{x + y\sqrt{2} : x, y \in \mathbb{Q}\}$. Show that the binary operation $\otimes$ defined by

$$(x + y\sqrt{2}) \otimes (z + w\sqrt{2}) = (xz + 2yw) + (xw + yz)\sqrt{2}$$

is an associative binary operation. (4 marks)

(d) Let R be a ring and $Z(R) = \{a \in R : ay = ya \text{ for all } y \in R\}$ i.e. $Z(R)$ is the center of R . Prove that $Z(R)$ is a subring of R . (4 marks)

(e) Let $(\mathbb{Z}_{18}, \oplus, \otimes)$ denote the ring of integers modulo 18 under addition and multiplication modulo 18. Let I be the *principal ideal* of $\mathbb{Z}_{18}$ generated by 3.

- (i) Determine all elements of I . (4 marks)
 - (ii) Construct all the elements of the *quotient ring* $\mathbb{Z}_{18}/I$. (4 marks)
- (f) Let A be a ring with identity $f: A \rightarrow B$ be a surjective ring homomorphism. Prove that B is also a ring with identity. (4 marks)

QUESTION TWO - (20 MARKS)

- (a) Distinguish between a *zero divisor* and a *unit* in an ring R . (4 marks)
- (b) Let R be a ring such that for any $a, b \in R$. Show that R is a commutative ring. (5 marks)
- (c) Let $\mathbb{Z}[i] = \{x + iy | x, y \in \mathbb{Z}\} \subset \mathbb{C}$ be the set of Gaussian integers. Determine all the units in $\mathbb{Z}[i]$. (5 marks)
- (d) Suppose that $\varphi: A \rightarrow B$ is a surjective ring homomorphism. Prove that
 - (i) if $I \subset A$ is a prime ideal then the image $\varphi(I)$ is a prime ideal of B , (3 marks)
 - (ii) if $J \subset B$ is a prime ideal then $\varphi^{-1}(J) = \{a \in A | \varphi(a) \in J\}$ is also a prime ideal. (3 marks)

QUESTION THREE (20 MARKS)

- (a) Explain the meaning of the following terms.
 - (i) a *subring* S in a ring R , (2 marks)
 - (ii) a *principal ideal* in a ring R . (2 marks)
- (b) Let $M_2(\mathbb{C})$ be the ring of all 2×2 matrices with complex entries. Consider the subset $T \subset M_2(\mathbb{C})$ given by

$$D = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

- i.e. T is the subset containing upper triangular matrices. Prove that T is a subring of $M_2(\mathbb{C})$. (5 marks)
- (c) Suppose that A and B are rings. Prove that if $\varphi: A \rightarrow B$ is a ring homomorphism then
 - (i) $\varphi(A)$ is a subring of B , (4 marks)
 - (ii) $\text{Ker } \varphi$ is an ideal of A , (4 marks)
 - (iii) if A is a commutative ring then $\varphi(A)$ is commutative. (3 marks)

QUESTION FOUR (20 MARKS)

- (a) Explain the meaning of the following terms (4 marks)

- (i) a *prime ideal* in a ring R ,
 - (ii) a *maximal ideal* in a ring R .
- (b) Suppose that U and V are ideals of a ring R . Prove that $U \cap V$ is also an ideal of R . (4 marks)
- (c) Let $S = \{m + n\sqrt[3]{2} : m, n \in \mathbb{Z}\} \subset \mathbb{R}$. Determine whether S is a subring of $\mathbb{R}$. (4 marks)
- (d) Let R be a ring and $f: \mathbb{Q} \rightarrow R$, $g: \mathbb{Q} \rightarrow R$ be ring homomorphisms such that $f(n) = g(n)$ for every $n \in \mathbb{Z}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Hint: Every element of $\mathbb{Q}$ can be written as $\frac{m}{n}$ for some $m, n \in \mathbb{Z}$ with $n \neq 0$. (8 marks)

QUESTION FIVE (20 MARKS)

- (a) Consider the binary operations of addition and multiplication defined respectively on $\mathbb{R}^2$ by:
- $$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \quad \text{and} \quad (x_1, x_2) \cdot (y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1).$$
- (i) Prove that $(\mathbb{R}^2, +, \cdot)$ is a ring. (6 marks)
 - (ii) Consider the map $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\psi(x, y) = x + y$. Determine if ψ a ring homomorphism. (4 marks)
- (b) Let $\mathbb{Q}[\sqrt{3}]$ denote the set $\{x + y\sqrt{3} : x, y \in \mathbb{Q}\}$. Define the binary operations $\oplus$ and $\otimes$ on $\mathbb{Q}[\sqrt{3}]$ as follows
- $$\begin{aligned} (x + y\sqrt{3}) \oplus (z + w\sqrt{3}) &= (x + y) + (y + w)\sqrt{3} \\ (x + y\sqrt{3}) \otimes (z + w\sqrt{3}) &= (xz + 3yw) + (xw + yz)\sqrt{3} \end{aligned}$$
- (i) Prove that $\mathbb{Q}[\sqrt{3}]$ is a subring of $\mathbb{R}$. (4 marks)
 - (ii) Determine if every non-zero elements of $\mathbb{Q}[\sqrt{3}]$ is invertible. (6 marks)