MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FIRST YEAR FIRST SEMESTER EXAMINATION FOR

MASTER OF SCIENCE (MATHEMATICAL MODELLING AND COMPUTATIONAL)

SMA 804: MODELING WITH ORDINARY DIFFERENTIAL EQUATIONS

DATE: 23/6/2021 **TIME: 2:00 – 5:00 PM**

INSTRUCTION: *Answer Question **ONE** which is compulsory and any other **TWO** Questions*

QUESTION ONE (20MARKS)

a) State any four steps of mathematical modeling (3 marks)

b) An entrepreneur wants to breed rainbow fish to sell to pet stores. He starts with a nice big aquarium and 30 fish, half of them male, half of them female. He wants to predict the number of fish after a number of days, to see how many he can sell.

i) Let's define t as the time. The symbol for the unit of t is $[t]$. Which unit for t is the most appropriate for our rainbow fish? (1 mark)

ii) Let us choose $P(t)$ as the function which describes the number of rainbow fish at time. For which word does the letter P stand? (1 mark)

iii) What is the initial condition for the rainbow fish? (1 mark)

iv) Assume that a rainbow fish naturally lives for approximately 1000 days and that the ages of the fish in the aquarium are mixed. The death rate d is the rate at which the fish die: per fish, how many fish die per day? (The unit d of the death rate is per day.) Give a value for the death rate d with three decimals. (2 marks)

v) Assume that one female rainbow fish lays eggs every 30 days and 42 of the eggs hatch to baby fish, half of them male, half female. On average, how many baby rainbow fish does one female rainbow fish produce per day? Give your answer correct to one decimal. (2 marks)

vi) Derive the corresponding differential equation using balance equation and taking into consideration the above scenarios from $(i)-(iv)$ to represent the population growth model for rainbow fish. Solve the resulting differential equation and validate this solution. In other words: does this solution represent a realistic solution to the problem? Give reason(s) (5 marks)

vii) Building on the mathematical model above, the aquarium owner expects to sell 20 rainbow fish every day. Derive the second mathematical model incorporating the removal. (5 marks)

viii) Investigate the solutions for new model for the problem and determine the stability of the equilibrium point(s) (5 marks)

ix) Using Euler method to determine how long it will take for the rainbow fish population to get into its equilibrium state? (5 marks)

QUESTION TWO (20MARKS)

The following data were obtained for the growth of a sheep population introduced into a new environment on the island of Tasmania.

t(years)	1814	1824	1834	1844	1854	1864
P(t)	125	275	830	1200	1750	1650

i) Make an estimate of M by graphing P(t). `(5 marks)

ii) Plot $\ln\left[\left(\frac{P}{M-P}\right)\right]$ against t. If a logistic curve seems reasonable, estimate rM and t^*

(15 marks)

QUESTION THREE (20 MARKS)

a) Consider the spreading of a highly communicable disease on an isolated island with population size N. A portion of the population travels abroad and returns to the island infected with the disease. You would like to predict the number of the people X who will have been infected by some time t. Consider the following model, where $k > 0$ is constant:

$$\frac{dX}{dt} = kX(N - X)$$

i) List major assumptions implicit in the preceding model. How reasonable are your assumptions? (3 marks)

ii) Graph dX/dt versus X (4 marks)

iii) Graph X versus t if the initial number of infections is $X_1 < N/2$. .Graph X versus $X_2 < N/2$. if the initial number of infections is (4 marks)

iv) Solve the model given earlier for X as a function of t (4 marks)

b) The autonomous differential equations represents growth population model.

$$\frac{dP}{dt} = 3P(1 - P)\left(P - \frac{1}{2}\right)$$

use the phase line analysis to sketch solution curves for $P(t)$, selecting different starting values $P(0)$, which equilibrium are stable, and which are unstable (5 marks)

QUESTION FOUR (20 MARKS)

The SEIR model consists of the following models equations:

$$\begin{aligned}\dot{S} &= -\beta SI + \lambda - \mu S \\ \dot{E} &= \beta SI - (\mu + k)E \\ \dot{I} &= kE - (\gamma + \mu)I \\ \dot{R} &= \gamma I - \mu R\end{aligned}$$

a) Sketch the flow diagram of the above model equations. (5 marks)

b) State what each of the following parameters β, λ, μ, k, γ represents. (5 marks)

c) Obtain the matrices F and V^{-1} for this model, hence show that the reproduction number is given by $R_0 = \dfrac{k\beta\lambda}{\mu(k+\mu)(\gamma+\mu)}$ (10 marks)

QUESTION FIVE (20 MARKS)

The following system represents the competition model

$$\frac{dx}{dt} = x(3-x) - 2xy$$
$$\frac{dy}{dt} = y(2-y) - xy$$

Where $x-$represents rabbits and $y-$represents sheep. Analyze the stability of the model.