MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

BACHELOR OF SCIENCE (MATHEMATICS), BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE), BACHELOR OF EDUCATION (SCIENCE) & BACHELOR OF EDUCATION (ARTS), FOURTH YEAR

SMA 433: PARTIAL DIFFERENTIAL EQUATIONS II

DATE: APRIL 2020 **TIME: 2 HRS**

INSTRUCTIONS: Answer Question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

a) State the second order linear partial differential equation (P.D.E.) in its general form. Hence name the three classical P.D.Es that are the canonical representations of this general 2nd order linear P.D.E. (5 marks)

b) Show that the set of functions $\left\{\cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}\right\}$ is orthogonal with respect to $w(x) = 1$ over the interval $[-L, L]$ (5 marks)

c) Write the Fourier Cosine series expansion of a function $f(x)$ over $[0, L]$. (3 marks)

d) Reduce the Laplace equation defined in a rectangular domain $D = \{(x, y) | 0 \le x \le a, 0 \le y \le b\}$ to a set of ordinary differential equations. (5 marks)

e) Given that $u = u(x, y)$, apply the change of variables $\theta = \theta(x, y)$, $\eta = \eta(x, y)$ to express the second derivatives u_y and u_{xy} in terms of the variables θ, η. (5 marks)

f) Under what assumptions on f, f', f'' does the following Fourier Sine transform hold:

$$\tilde{F}_S\{f''(x)\} = \int_0^\infty f''(x)\sin\alpha x dx = -\alpha^2 F(\alpha) + \alpha f(0)$$ (5 marks)

QUESTION TWO (20 MARKS)

a) Determine the region in the xy-plane where the partial differential equation

$u_{xx} + yu_{yy} + \dfrac{u_y}{2} = 0$ is elliptic, and hence obtain the associated canonical form.(10 marks)

b) The displacement of a semi-infinite elastic beam is described

$$u_{tt} = u_{xx}\ ,\ x > 0, t > 0$$

by $u(0,t) = t$, $\lim\limits_{x\to\infty} \dfrac{\partial u}{\partial x} = 0$, $t > 0$. Solve for $u(x,t)$ by the Laplace transform method.

$$u(x,0) = 0,\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\ t > 0$$

(10 marks)

QUESTION THREE (20 MARKS)

a) Use the appropriate Fourier transform to solve the given boundary-value

$$u_{xx} + u_{yy} = 0,\ 0 < x < \pi,\ y > 0$$

problem $u(0, y) = f(y)$, $\left.\dfrac{\partial u}{\partial x}\right|_{x=\pi} = 0,\ y > 0$ (13 marks)

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\ 0 < x < \pi$$

b) The Laplace transform of a function $u = u(x,t)$ is $L\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)\,dt$. Determine the Laplace transform of $\dfrac{\partial^2 u}{\partial t^2}$. (7 marks)

QUESTION FOUR (20 MARKS)

a) Apply the separation of variables method to solve the following initial boundary value problem for temperature in a homogeneous isotropic rod with insulated sides and no internal heat generation:

$$u_t = ku_{xx}\ ,\ 0 < x < \pi, t > 0$$
$$u_x(0,t) = u_x(\pi,t) = 0,\ t > 0$$
$$u(x,0) = x,\ 0 < x < \pi$$

The given boundary conditions imply that both ends of the rod are also insulated.

(15 marks)

b) Let $f(x)$, $g(x)$ be functions whose Fourier transform is $F(w)$, $G(w)$ respectively. Let $h(x)$ be the convolution of $f(x)$, $g(x)$ such that its Fourier transform $H(w) = F(w)G(w)$.

Show that $h(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} g(\tau) f(x-\tau)\,d\tau$ (3 marks)

c) The boundary value problem
$$-(p(x)y'(x))' + q(x)y(x) = f(x),\ 0 < x < 1$$
$$y(0) - h_0 y'(0) = 0$$
$$y(1) - h_1 y'(1) = 0$$
can be solved by determining a function $G(x;s)$ so that the solution is $y(x) = \int_0^1 G(x;s) f(s)ds$.

i) What is the name given to the given boundary value problems? (1 marks)

ii) State the conditions that must be satisfied for function $G(x;s)$ to be called a Green's function for the given boundary value problem. (2 marks)

QUESTION FIVE (20 MARKS)

a) Find the characteristics of the equation $u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 6u_y = 9(2x - y)$ and reduce it to the appropriate canonical form. (11 marks)

b) Given the function $f(x) = x,\ 0 < x < 4,$ such that $f(x+4) = f(x)$, determine the Fourier series expansion of the function. (9 marks)